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RESEARCH ARTICLE

Dirichlet Average of Wright function and Fractional Derivative

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Abstract:
In this paper we establish a relation Dirichlet average of $W_{\alpha,\beta}(x)$unction, using fractional derivative.

Key Words: Dirichlet average, $W_{\alpha,\beta}(x)$and Fractional calculus operators.

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1.Introduction

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions likex^t,e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n,e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

In this paper the Dirichlet average of Hyper-geometric function has been obtained.

1. Definitions:

We give blew some of the definitions which are necessary in the preparation of this paper.

1.1 Standard Simplex in $R^n, n \geq 1$:

We denote the standard simplex in R^n, $n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_{1,}u_2, \dots \dots .. u_n) \ : \ u_1 \geq 0, \dots \dots \dots . u_n \geq 0, \ u_1 + u_2 + \cdots \dots \dots + u_n \leq 1\} \qquad (2.1.1)$$

1.2 Dirichlet measure:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by E[1].

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots \dots \dots \dots . u_{k-1}^{b_{k-1}-1} (1 - u_1 - \cdots \dots \dots \dots - u_{k-1}) b_k^{-1} du_1 \dots \dots \dots \dots . du_{k-1} \qquad (2.2.1)$$

Will be called a Dirichlet measure.
Here

$$B(b) = B(b1, \dots \dots \dots . bk) = \frac{\Gamma(b_1) \dots \dots \dots \dots \dots . . \Gamma(b_k)}{\Gamma(b_1 + \cdots \dots \dots . . + b_k)},$$
$$C_{>} = \{z \in z : z \neq 0, |ph\, z| < \pi/_2\},$$

Open right half plane and $C_{>}$k is the k^{th} Cartesian power of $C_{>}$

1.3 Dirichlet Average[1, p.75]:

Let Ω be the convex set in $C_{>}$, let $z = (z_1, \dots \dots \dots , z_k) \in \Omega^{\mathrm{k}}, \mathrm{k} \geq 2$ and let $u.z$ be a convex combination of $z_1, \dots \dots \dots , z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b,z) = \int_E f(u.z)d\,\mu_b(u) \tag{2.3.1}$$

We shall call F the Dirichlet measure of f with variables
$z = (z_1, \dots \dots \dots, z_k)$ and parameters $b = (b_1, \dots \dots \dots . b_k)$.
Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \cdots \dots \dots . -u_{k-1} \tag{2.3.2}$$

If $k = 1$, define $F(b,z) = f(z)$.

1.4 Fractional Derivative [8, p.181]:

The concept of fractional derivative with respect to an arbitrary function has been used by Erdelyi[8]. The most common definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \tag{2.4.1}$$

Where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$.

2.5 The wright function $W_{\alpha,\beta}(z)$:

The Wright function, that denote by $W_{\alpha,\beta}(z)$ is so named in honors of E. Maitland Wright, the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions, see [Wright (1933); 1935a; 1935b].The function is defined by the series representation, convergent in the whole z-complex plane,

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\alpha n + \beta)}, \ \alpha > -1, \quad \beta \in C \tag{2.5.1}$$

so $W_{\alpha,\beta}(z)$ is an entire function. Originally Wright assumed $\alpha > 0$, and, only in 1940, he considered $-1 < \alpha < 0$, see [Wright 1940]. We note that in the handbook of the Bateman Project [Erdelyi et al. Vol. 3, Ch. 18], presumably for a misprint, α is restricted to be non negative.

Equivalence:

In this section we shall show the equivalence of single Dirichlet average of $W_{\alpha,\beta}(x)$function $(k = 2)$ with the fractional derivative i.e.

$$S(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma\beta} (x-y)^{1-\beta-\beta'} D_{x-y}^{-\beta'} W_{\alpha,\beta}(x)(x-y)^{\beta-1} \tag{3.1}$$

Proof:

$$S(\beta, \beta'; x, y) = \sum_{n=o}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{1}{n!} R_n(\beta, \beta'; x, y)$$

$$= \sum_{n=o}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{\Gamma(\beta + \beta')}{\Gamma\beta\,\Gamma\beta'} \int_0^1 [ux + (1-u)y]^n \, u^{\beta-1}(1-u)^{\beta'-1} du$$

Putting $u(x - y) = t$, we have,

$$= \sum_{n=o}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{\Gamma(\beta + \beta')}{\Gamma\beta\,\Gamma\beta'} \int_0^{x-y} [t + y]^n \left(\frac{t}{x-y}\right)^{\beta-1} \left(1 - \frac{t}{x-y}\right)^{\beta'-1} \frac{dt}{x-y}$$

On changing the order of integration and summation, we have

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta + \beta')}{\Gamma\beta\,\Gamma\beta'} \int_0^{x-y} \sum_{n=o}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} [t + y]^n \, (t)^{\beta-1}(x-y-t)^{\beta'-1} dt$$

Or

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\,\Gamma\beta'} \int_0^{x-y} W_{\alpha,\beta}(x)\,(t)^{\beta-1}(x-y-t)^{\beta'-1}dt$$

Hence by the definition of fractional derivative, we get

$$S(\beta,\beta';x,y) = (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta} D_{x-y}^{-\beta'} W_{\alpha,\beta}(x)(x-y)^{\beta-1}$$

This completes the Analysis.

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