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**RESEARCH ARTICLE**

# FRACTIONAL q-DERIVATIVE OF S- FUNCTION

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***Abstract:***
This paper is devoted to fractional q-derivative of special functions. To begin with the theorem on term by term q-fractional differentiation has been derived. The result is an extension of an earlier result due to Yadav and Purohit [8] and Sharma and Jain [9]. As a special case, of fractional q-differentiation of S-Function has been obtained.

***Key Words:*** Fractional integral and derivative operators, Fractional q-derivative, S-Function and Special functions.

**Mathematics Subject Classification**— Primary33A30, Secondary 33A25, 83C99

**Definition:**

**1.1. q-Analogue of Differential Operator**

Al-Salam [3], has given the q-analogue of differential operator as

$$D_q f(x) = \frac{f(xq) - f(x)}{x(q-1)} \tag{1.1}$$

This is an inverse of the q-integral operator defined as

$$\int_x^\infty f(t)\, d(t{:}q) = x(1-q) \sum_{k=1}^{\infty} q^{-k} f(xq^{-k}) \tag{1.2}$$

WHERE $0 < |q| < 1$

**1.2. FRACTIONAL Q-DERIVATIVE OF ORDER $\alpha$:**

THE FRACTIONAL Q-DERIVATIVE OF ORDER $\alpha$ IS DEFINED AS

$$D_{x,q}^{\alpha} f(x) = \frac{1}{\boldsymbol{\Gamma}_{\mathrm{q}}(-\alpha)} \int_0^x (x - yq)_{-\alpha-1} f(y) d(y;q) \tag{1.2.1}$$

WHERE RE $(\alpha) < 0$

AS A PARTICULAR CASE OF (3), WE HAVE

$$D_{x,q}^{\alpha} x^{\mu-1} = \frac{\boldsymbol{\Gamma}_{\mathrm{q}}(\mu)}{\boldsymbol{\Gamma}_{\mathrm{q}}(\mu-\alpha)} x^{\mu-\alpha-1} \tag{1.2.2}$$

**1.3 S-Function:**

This function introduced by the author is defined as follows:

$${}^{a}S_{p,q}^{\alpha,\beta}\left(a_1 \,....\, a_p;, b_1 \,....\, b_q; x\right) = \sum_{n=o}^{\infty} \frac{(a_1)_n \,.....\, (a_p)_n}{(b_1)_n \,.....\, (b_q)_n} k \frac{a^n x^n}{\Gamma(\alpha n + \beta + 1) n!} \tag{1.3.1}$$

Here, $p$ upper parameters $a_{1,} a_{2,} \ldots\ldots a_p$ and $q$ lower parameters $b_{1,}\ b_{2,} \ldots\ldots b_q,\ \alpha, \beta \epsilon C,\ R(\alpha) > 0$, $R(\beta) > 0$ and $(a_j)_k\ (b_j)_k$ are pochammer symbols and K is constant. The function (3) is defined when none of the denominator parameters $b_j s,\ j = 1,2, \ldots \ldots \ldots q$ is a negative integer or zero. If any parameter $a_j$ is negative then the function (3) terminates into a polynomial in x. By using ratio test, it is evident that function (3) is convergent for all x, when $q \geq p$, it is convergent for $|x| < 1$ when $p = q + 1$, divergent when $p > q + 1$. In some cases the series is convergent for $x = 1,\ x = -1$. Let us consider take,

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$$

when $p = q + 1$, the series is absolutely convergent for $|x| = 1$ if $R(\beta) < 0$, convergent for $x = -1$, if $0 \leq R(\beta) < 1$ and divergent for $|x| = 1$, if $1 \leq R(\beta)$.

## 2. Main Results

In this section we drive the results on term by term q-fractional differentiation of a power series. As particular case we will the fractional q-differentiation of the S-Function and exponential series.

Theorem 1: If the S-Function ${}^{a}M_{p,q}^{\alpha,\beta}(z)$converges absolutely for $|q| < \rho$ then

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k}{(b_1)_k\ \ldots\ldots(b_q)_k}k\frac{a^k}{\Gamma(\alpha k+\beta+1)k!}\right\}$$

$$=\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k\,a^k}{(b_1)_k\ \ldots\ldots(b_q)_k\Gamma(\alpha k+\beta+1)k!}kD_{z,q}^{\mu}z^{k+\lambda-1} \qquad (2.1)$$

Where Re $(\lambda) > 0$, Re $(\mu) < 0$, $0 < |q| < 1$

Proof: Starting From the left side and using equation (1.2.1), we have

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k}{(b_1)_k\ \ldots\ldots(b_q)_k}\mathrm{k}\frac{a^kz^k}{\Gamma(\alpha k+\beta+1)k!}\right\}$$

$$=\frac{1}{\boldsymbol{\Gamma}_{\mathrm{q}}(-\mu)}\int_0^z(z-yq)_{-\mu-1}\,y^{\lambda-1}\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k}{(b_1)_k\ \ldots\ldots(b_q)_k}\mathrm{k}\frac{a^ky^k}{\Gamma(\alpha k+\beta+1)k!}d(y;q)$$

$$=\frac{z^{\lambda-\mu-1}}{\boldsymbol{\Gamma}_{\mathrm{q}}(-\mu)}\int_0^1(1-tq)_{-\mu-1}\,t^{\lambda-1}\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k}{(b_1)_k\ \ldots\ldots(b_q)_k}\mathrm{k}\frac{a^kt^k}{\Gamma(\alpha k+\beta+1)k!}d(t;q) \qquad (2.2)$$

Now the following observation are made

$(i)$ $\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k}{(b_1)_k\ \ldots\ldots(b_q)_k}k\frac{a^kt^k}{\Gamma(\alpha k+\beta+1)k!}$ converges absolutely and therefore uniformly on domain of x over the region of integration.

$(ii)$ $\int_0^1\left|(1-tq)_{-\mu-1}t^{\lambda-1}\right|\ d(t;q)$ is convergent,

Provided Re $(\lambda) > 0$, Re $(\mu) < 0$, $0 < |q| < 1$

Therefore the order of integration and summation can be interchanged in (2.2) to obtain.

$$=\frac{z^{\lambda-\mu-1}}{\boldsymbol{\Gamma}_{\mathrm{q}}(-\mu)}\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k}{(b_1)_k\ \ldots\ldots(b_q)_k}k\frac{a^k}{\Gamma(\alpha k+\beta+1)k!}\int_0^1(1-tq)_{-\mu-1}\,t^{\lambda+k-1}d(t;q)$$

$$=\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k}{(b_1)_k\ \ldots\ldots(b_q)_k}k\frac{a^k}{\Gamma(\alpha k+\beta+1)k!}D_{z,q}^{\mu}z^{k+\lambda-1}$$

Hence the statement (2.1) is proved.

## 3. Some special cases:

(i) If we take $= 0, \beta = 0$ , Constant k=1 in equation (2.1) it becomes the fractional q-derivative of power series.

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k}{(b_1)_k\ \ldots\ldots(b_q)_k}a^kz^k\right\}=\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots\ldots(a_p)_k}{(b_1)_k\ \ldots\ldots(b_q)_k}a^kD_{z,q}^{\mu}\{z^{k+\lambda-1}\} \qquad (3.1)$$

This equation (3.1) is known result given by Yadav and Purohit [8] and Ali, Jain and Sharma [9].

(ii) When $\alpha = 1, \beta = 0, a = 1$ , Constant k=1 and no upper or lower parameter in(2.1), we have

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}\sum_{k=o}^{\infty}\frac{z^k}{\Gamma(k+1)}\right\}=\sum_{k=o}^{\infty}\frac{1}{k!}D_{z,q}^{\mu}\left\{z^{k+\lambda-1}\right\} \tag{3.2}$$

EQUIVALENTLY,

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}e^{z}\right\}=\sum_{k=o}^{\infty}\frac{1}{k!}D_{z,q}^{\mu}\left\{z^{k+\lambda-1}\right\} \tag{3.3}$$

Thus the equation reduces to fractional q-derivative of exponential function.

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