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RESEARCH ARTICLE

Dirichlet Average of J-Function and Fractional Derivative

Manoj Sharma
Department of Mathematics RJIT, BSF Academy, Tekanpur.

Abstract:
In this paper we establish a relation Dirichlet average of **J-Function**, using fractional derivative.

Key Words: Dirichlet average, J-Function, fractional derivative and Fractional calculus operators.

Mathematics Subject Classification: 26A33, 33A30, 33A25 and 83C99.

1. Introduction:

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions likex^t,e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n,e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

In this paper the Dirichlet average of Hyper-geometric function has been obtained.

2. Definitions:

We give blew some of the definitions which are necessary in the preparation of this paper.

2.1 Standard Simplex in $R^n, n \geq 1$:

We denote the standard simplex in R^n, $n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_{1,}u_2, \dots \dots . . u_n) \; : \; u_1 \geq 0, \dots \dots \dots . u_n \geq 0, \; u_1 + u_2 + \cdots \dots \dots + u_n \leq 1\} \qquad (2.1.1)$$

2.2 Dirichlet measure:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by $E[1]$.

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots \dots \dots \dots \dots . u_{k-1}^{b_{k-1}-1}(1 - u_1 - \cdots \dots \dots \dots - u_{k-1}) b_k^{-1} du_1 \dots \dots \dots \dots . du_{k-1} \qquad (2.2.1)$$

Will be called a Dirichlet measure.
Here

$$B(b) = B(b1, \dots \dots \dots . bk) = \frac{\Gamma(b_1) \dots \dots \dots \dots \dots . . \Gamma(b_k)}{\Gamma(b_1 + \cdots \dots \dots . . + b_k)},$$

$$C_> = \{z \in z : z \neq 0, |ph\, z| < {}^{\pi}/_{2}\},$$

Open right half plane and $C_>$k is the k^{th} Cartesian power of $C_>$

2.3 Dirichlet Average[1, p.75]:

Let Ω be the convex set in $C_>$, let $z = (z_1, \dots \dots \dots, z_k) \in \Omega^{\mathrm{k}}, \mathrm{k} \geq 2$ and let $u.z$ be a convex combination of $z_1, \dots \dots \dots, z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b,z)=\int_E f(u.z)d\,\mu_b(u) \tag{2.3.1}$$

We shall call F the Dirichlet measure of f with variables
$z=(z_1,\dots\dots\dots,z_k)$ and parameters $b=(b_1,\dots\dots\dots.b_k)$.
Here

$$u.z=\sum_{i=1}^{k}u_i z_i \text{ and } u_k=1-u_1-\cdots\dots\dots.-u_{k-1} \tag{2.3.2}$$

If $k=1$, define $F(b,z)=f(z)$.

2.4 Fractional Derivative [8, p.181]:

The concept of fractional derivative with respect to an arbitrary function has been used by Erdelyi[8]. The most common definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^{\alpha}F(z)=\frac{1}{\Gamma(-\alpha)}\int_0^z F(t)(z-t)^{-\alpha-1}dt \tag{2.4.1}$$

Where $Re(\alpha)<0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x=0$.

3.1 J-Function:

This function introduced by the author is defined as follows:

$${}^{a}J_{p,q}^{\alpha,\beta}\left(a_1\ .\ .\ .\ .a_p;,b_1\ .\ .\ .\ .b_q;x\ \right)=\sum_{n=o}^{\infty}\frac{(a_1)_n\ .\ .\ .\ .\ .(a_p)_n}{(b_1)_n\ .\ .\ .\ .\ .(b_q)_n}\frac{a^n\ b^n n\,x^n}{\Gamma(\alpha n+\beta+1)n!} \tag{3.1}$$

Here, p upper parameters $a_{1,}a_{2,}\ ......\ a_p$ and q lower parameters $b_{1,}\ b_{2,}\ .....b_q,\ \alpha,\beta\epsilon C,\ R(\alpha)>0$, $R(\beta)>0$ and $(a_j)_k\ (b_j)_k$ are pochammer symbols and a,b and n is constant. The function (1.3.1) is defined when none of the denominator parameters $b_j s,\ j=1,2,\dots\dots\dots q$ is a negative integer or zero. If any parameter a_j is negative then the function (1.3.1) terminates into a polynomial in x. By using ratio test, it is evident that function (1.3.1) is convergent for all x, when $q\geq p$, it is convergent for $|x|<1$ when $p=q+1$, divergent when $p>q+1$. In some cases the series is convergent for $x=1,\ x=-1$. Let us consider take,

$$\beta=\sum_{j=1}^{p}a_j-\sum_{j=1}^{q}b_j$$

when $p=q+1$, the series is absolutely convergent for $|x|=1$ if $R(\beta)<0$, convergent for $x=-1$, if $0\leq R(\beta)<1$ and divergent for $|x|=1$, if $1\leq R(\beta)$.

3.2 Equivalence:

In this section we shall show the equivalence of single Dirichlet average of **J-Function** $(k=2)$ with the fractional derivative i.e.

$$S(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma\beta}(x-y)^{1-\beta-\beta'}D_{x-y}^{-\beta'}\,{}^{a}J_{p,q}^{\alpha,\beta}(x)(x-y)^{\beta-1} \tag{3.2}$$

Proof:

$$S(\beta,\beta';x,y)=\sum_{n=o}^{\infty}\frac{(a_1)_n\ .\ .\ .\ .\ .(a_p)_n}{(b_1)_n\ .\ .\ .\ .\ .(b_q)_n}\frac{a^n\ b^n n}{\Gamma(\alpha n+\beta+1)}R_n(\beta,\beta';x,y)$$

$$=\sum_{n=o}^{\infty}\frac{(a_1)_n\ .\ .\ .\ .\ .(a_p)_n}{(b_1)_n\ .\ .\ .\ .\ .(b_q)_n}\frac{a^n\ b^n n}{\Gamma(\alpha n+\beta+1)}\frac{\Gamma(\beta+\beta')}{\Gamma\beta\,\Gamma\beta'}\int_0^1[ux+(1-u)y]^n\,u^{\beta-1}(1-u)^{\beta'-1}du$$

Putting $u(x-y)=t$, we have,

$$=\sum_{n=o}^{\infty}\frac{(a_1)_n\ .\ .\ .\ .\ .(a_p)_n}{(b_1)_n\ .\ .\ .\ .\ .(b_q)_n}\frac{a^n\ b^n n}{\Gamma(\alpha n+\beta+1)}\frac{\Gamma(\beta+\beta')}{\Gamma\beta\,\Gamma\beta'}\int_0^{x-y}[t+y]^n\left(\frac{t}{x-y}\right)^{\beta-1}\left(1-\frac{t}{x-y}\right)^{\beta'-1}\frac{dt}{x-y}$$

On changing the order of integration and summation, we have

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\,\Gamma\beta'} \int_0^{x-y} \sum_{n=o}^{\infty} \frac{(a_1)_n \ \ldots\ldots (a_p)_n}{(b_1)_n \ \ldots\ldots (b_q)_n} \frac{a^n\, b^n n}{\Gamma(\alpha n+\beta+1)} [t+y]^n\ (t)^{\beta-1}(x-y-t)^{\beta'-1} dt$$

Or

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\,\Gamma\beta'} \int_0^{x-y} {}^{a}J_{p,q}^{\alpha,\beta}(x)\ (t)^{\beta-1}(x-y-t)^{\beta'-1} dt$$

Hence by the definition of fractional derivative, we get

$$S(\beta,\beta';x,y) = (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta} D_{x-y}^{-\beta'}\, {}^{a}J_{p,q}^{\alpha,\beta}(x)(x-y)^{\beta-1}$$

This completes the Analysis.

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