ISSN 2348 – 0319 International Journal of Innovative and Applied Research (2013), Volume 1, Issue (2): 75- 78

ISSN 2348 - 0319

Journal home page: http://www.journalijiar.com

INTERNATIONAL JOURNAL OF INNOVATIVE AND APPLIED RESEARCH

RESEARCH ARTICLE

APPLICATION OF FRACTIONAL CALCULUS IN PHYSICAL EQUATION

[1]Manoj Sharma, [2]Renu Jain

1. Deptt. of Mathematics, RJIT, BSF, Tekanpur.
2. SOMAAS, Jiwaji University, Gwalior.

Abstract:

This paper takes up applications of fractional calculus to the problem of physics. We derive a closed form solution of fractional diffusion equation in terms of sums of H-functions.

1. A Closed Form Solution of Fractional Diffusion Equation:

The Fractional Diffusion Equation is given as

$$ {}_0D_t^{\alpha} N(x,t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\delta(x) = c^{\alpha}\ \frac{\partial^2}{\partial \mathrm{x}^2} N(x,t), \ t > 0,\ -\infty < x < \infty \qquad ...(1) $$

with the initial condition

$$ {}_0D_t^{\alpha-1} N(x,t)\Big|_{t=0} = \phi(x), \quad \lim_{\mathrm{x}\to\pm\infty} N(x,t) = 0 \qquad ...(2) $$

where $n = [\mathrm{Re}(\alpha)] + 1,\ c^{\alpha}$ is a diffusion constant and $\delta(x)$ is Dirac's delta function. Saxena, Mathai and Haubold [6] have obtained solution of (1) taking a special case $\phi(x) = 0$, we have obtained solution for more general initial conditions from which result in [6] follow as a particular case. The solution of (1) can be put in the following form.

$$ N(x,t) = \int_{-\infty}^{\infty} G(x-\rho,t)\phi(\rho)d\rho + \frac{1}{|x|} H_{2,2}^{2,0}\left[\frac{|x|^2}{c^{\alpha}t^{\alpha}} \middle| \begin{matrix} (1,\alpha), & (1,1) \\ (1,2), & (1,1) \end{matrix} \right] \qquad ...(3) $$

where $$ G(x,t) = \frac{\pi}{|x|^{\alpha}} H_{3,3}^{2,1}\left[\frac{|x|^{\alpha}}{c^{\alpha}k^2} \middle| \begin{matrix} (1,1),(\alpha,\alpha),(\frac{1}{2}+\frac{\alpha}{2},\frac{\alpha}{2}) \\ (\alpha,\alpha),(1,1),(\frac{1}{2}+\frac{\alpha}{2},\frac{\alpha}{2}) \end{matrix} \right] $$

Proof: In order to derive the solution of (1) we introduce the Laplace-Fourier transform in the form

$$N^*(k,s)=\int_0^{\infty}\int_{-\infty}^{\infty} e^{-st+ikx}\ N(x,t)\,dx\,dt \qquad ...(4)$$

Applying the Fourier transform with respect to the space variable x and Laplace transform with respect to the time variable t to equation (1) we get

$$s^{\alpha}N^*(k,s)-{}_0D_t^{\alpha-1}N^*(\mathrm{k,t})\Big|_{t=0}-s^{\alpha-1}=-c^{\alpha}k^2N^*(k,s) \qquad ...(5)$$

By using initial condition (2) we find that

$$s^{\alpha}N^*(k,s)-\phi(k)-s^{\alpha-1}=-c^{\alpha}k^2N^*(k,s) \qquad ...(6)$$

Solving for N* (k, s) this gives

$$\mathrm{N^*(k,s)}\,[\mathrm{s}^{\alpha}+c^{\alpha}k^2]=\phi(\mathrm{k})+\mathrm{s}^{\alpha-1} \qquad ...(7)$$

$$N^*(k,s)=\frac{\phi(\mathrm{k})}{\mathrm{s}^{\alpha}+c^{\alpha}k^2}+\frac{\mathrm{s}^{\alpha-1}}{\mathrm{s}^{\alpha}+c^{\alpha}k^2} \qquad ...(8)$$

$$=\phi(\mathrm{k})\mathrm{s}^{-\alpha}[1+c^{\alpha}k^2s^{-\alpha}]^{-1}+\mathrm{s}^{-1}[1+c^{\alpha}k^2s^{-\alpha}]^{-1} \qquad ...(9)$$

By applying binomial expansion in equation (9), we obtain

$$\mathrm{N^*(k,s)}=\phi(k)\sum_{r=0}^{\infty}\frac{(-1)^r c^{\alpha r}k^{2r}s^{(-\alpha r-\alpha)}}{r!}+\sum_{r=0}^{\infty}\frac{(-1)^r c^{\alpha r}k^{2r}s^{(-\alpha r-1)}}{r!} \qquad ...(10)$$

Applying the inverse Laplace transform given by the relation

$$L^{-1}\{s^{-\rho}\}=\frac{t^{\rho-1}}{\Gamma(\rho)}\ ,\ \mathrm{Re}(\rho)>0 \qquad ...(11)$$

equation (10) becomes

$$N^*(k,t)=\phi(\mathrm{k})\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}[-c^{\alpha}k^2t^{\alpha}]+E_{\alpha}(-c^{\alpha}k^2t^{\alpha}) \qquad ...(12)$$

where $E_{\alpha,\alpha}(x)=\sum_{r=0}^{\infty}\frac{x^r}{\Gamma(\alpha r+\alpha)}$ is a generalization of Mittag-Leffler function.Equation (12) can be expressed in terms of the H-function [7]. Hence we get

$$N*(k,t)=\phi(k)t^{\alpha-1}H_{1,2}^{1,1}\left[c^{\alpha}k^{2}t^{\alpha}\left|\begin{matrix}(0,1)\\(0,1),(1-\alpha,\alpha)\end{matrix}\right.\right]+$$

$$H_{1,2}^{1,1}\left[c^{\alpha}k^{2}t^{\alpha}\left|\begin{matrix}(0,1)\\(0,1),(0,\alpha)\end{matrix}\right.\right] \qquad ...(13)$$

Next we take inverse Fourier transform in (13), using convolution of integral and keeping in view that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}f(k)dk=\frac{1}{\pi}\int_{0}^{\infty}f(k)\cos(kx)dk \qquad ...(14)$$

and that the value of inverse Fourier transform of H-function is given by

$$\int_{0}^{\infty}t^{\rho-1}\cos(kt)H_{p,q}^{m,n}\left[at^{\mu}\left|\begin{matrix}(a_p,Ap\\(b_q,B_q)\end{matrix}\right.\right]dt$$

$$=(\pi/k^{\rho})H_{q+1,p+2}^{n+1,m}\left[\frac{k^{\mu}}{a}\left|\begin{matrix}(1-b_q,B_q),(\frac{1}{2}+\frac{\rho}{2},\frac{\mu}{2})\\(\rho,\mu),(1-a_p,A_p),(\frac{1}{2}+\frac{\rho}{2},\frac{\mu}{2})\end{matrix}\right.\right],$$

$$...(15)$$

where $\mathrm{Re}\left[\rho+\mu_{1\le j\le m}^{\min}\left(\frac{b_j}{B_j}\right)\right]>0,\ \mu>\mathrm{Re}\left[\rho+\mu_{1\le j\le n}^{\max}\frac{(a_j-1)}{(A_j)}\right]$

$$<1,\left|\arg a\right|<(\pi\Theta/2),\Theta_1>0,$$

$$\Theta_1=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j. \qquad ...(16)$$

Thus we obtained the desired result (3).

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