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RESEARCH ARTICLE

FRACTIONAL q-DERIVATIVE OF J- FUNCTION

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Abstract:
This paper is devoted to fractional q-derivative of special functions. To begin with the theorem on term by term q-fractional differentiation has been derived. The result is an extension of an earlier result due to Yadav and Purohit [8] and Sharma and Jain [9]. As a special case, of fractional q-differentiation of J-Function has been obtained.

Key Words: Fractional integral and derivative operators, Fractional q-derivative, J-Function and Special functions

Definition:

1.1. q-Analogue of Differential Operator

Al-Salam [3], has given the q-analogue of differential operator as

$$D_q f(x) = \frac{f(xq) - f(x)}{x(q-1)} \tag{1.1}$$

This is an inverse of the q-integral operator defined as

$$\int_x^\infty f(t)\, d(t:q) = x(1-q) \sum_{k=1}^{\infty} q^{-k} f(xq^{-k}) \tag{1.2}$$

WHERE $0 < |q| < 1$

1.2. FRACTIONAL Q-DERIVATIVE OF ORDER α:

THE FRACTIONAL Q-DERIVATIVE OF ORDER α IS DEFINED AS

$$D_{x,q}^{\alpha} f(x) = \frac{1}{\mathbf{\Gamma}_q(-\alpha)} \int_0^x (x - yq)_{-\alpha-1} f(y) d(y;q) \tag{1.2.1}$$

WHERE RE $(\alpha) < 0$

AS A PARTICULAR CASE OF (3), WE HAVE

$$D_{x,q}^{\alpha} x^{\mu-1} = \frac{\mathbf{\Gamma}_q(\mu)}{\mathbf{\Gamma}_q(\mu-\alpha)} x^{\mu-\alpha-1} \tag{1.2.2}$$

1.3 J-Function:

This function introduced by the author is defined as follows:

$${}^{a}J_{p,q}^{\alpha,\beta}\left(a_1 \, . \, . \, . \, . \, a_p; , b_1 \, . \, . \, . \, . \, b_q; x\right) = \sum_{n=o}^{\infty} \frac{(a_1)_n \, . \, . \, . \, . \, . \, (a_p)_n}{(b_1)_n \, . \, . \, . \, . \, . \, (b_q)_n} \frac{a^n\, b^n n\, x^n}{\Gamma(\alpha n + \beta + 1) n!} \tag{1.3.1}$$

Here, p upper parameters $a_{1,} a_{2,} \ldots\ldots a_p$ and q lower parameters $b_{1,}\ b_{2,} \ldots\ldots b_q,\ \alpha, \beta \epsilon C,\ R(\alpha) > 0$, $R(\beta) > 0$ and $(a_j)_k\ (b_j)_k$ are pochammer symbols and a,b and n is constant. The function (1.3.1) is defined when none of the denominator parameters $b_j s,\ j = 1,2, \ldots \ldots \ldots q$ is a negative integer or zero. If any parameter a_j is negative then the function (1.3.1) terminates into a polynomial in x. By using ratio test, it is evident that function (1.3.1) is convergent for all x, when $q \geq p$, it is convergent for $|x| < 1$ when $p = q + 1$, divergent when $p > q + 1$. In some cases the series is convergent for $x = 1,\ x = -1$. Let us consider take,

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$$

when $p = q + 1$, the series is absolutely convergent for $|x| = 1$ if $R(\beta) < 0$, convergent for $x = -1$, if $0 \leq R(\beta) < 1$ and divergent for $|x| = 1$, if $1 \leq R(\beta)$.

2. MAIN RESULTS

IN THIS SECTION WE DRIVE THE RESULTS ON TERM BY TERM Q-FRACTIONAL DIFFERENTIATION OF A POWER SERIES. AS PARTICULAR CASE WE WILL THE FRACTIONAL Q-DIFFERENTIATION OF THE S-FUNCTION AND EXPONENTIAL SERIES.

THEOREM 1: IF THE S-Function ${}^{a}J_{p,q}^{\alpha,\beta}(z)$converges absolutely for $|q| < \rho$ THEN

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}\sum_{n=o}^{\infty}\frac{(a_1)_n \ \ldots\ldots (a_p)_n}{(b_1)_n \ \ldots\ldots (b_q)_n}\frac{a^n\, b^n n\, x^n}{\Gamma(\alpha n+\beta+1)n!}\right\}$$

$$= \sum_{n=o}^{\infty}\frac{(a_1)_n \ \ldots\ldots (a_p)_n}{(b_1)_n \ \ldots\ldots (b_q)_n}\frac{a^n\, b^n n\, x^n}{\Gamma(\alpha n+\beta+1)n!} D_{z,q}^{\mu} z^{n+\lambda-1} \qquad (2.1)$$

Where RE $(\lambda) > 0$, RE $(\mu) < 0$, $0 < |q| < 1$

PROOF: STARTING FROM THE LEFT SIDE AND USING EQUATION (1.2.1), WE HAVE

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}\sum_{n=o}^{\infty}\frac{(a_1)_n \ \ldots\ldots (a_p)_n}{(b_1)_n \ \ldots\ldots (b_q)_n}\frac{a^n\, b^n k\, x^n}{\Gamma(\alpha n+\beta+1)n!}\right\}$$

$$= \frac{1}{\mathbf{\Gamma}_{\mathrm{q}}(-\mu)}\int_0^z (z-yq)_{-\mu-1}\, y^{\lambda-1}\sum_{k=o}^{\infty}\frac{(a_1)_k \ \ldots\ldots (a_p)_k}{(b_1)_k \ \ldots\ldots (b_q)_k}\frac{a^k b^k y^k k}{\Gamma(\alpha k+\beta+1)k!} d(y;q)$$

$$= \frac{z^{\lambda-\mu-1}}{\mathbf{\Gamma}_{\mathrm{q}}(-\mu)}\int_0^1 (1-tq)_{-\mu-1}\, t^{\lambda-1}\sum_{k=o}^{\infty}\frac{(a_1)_k \ \ldots\ldots (a_p)_k}{(b_1)_k \ \ldots\ldots (b_q)_k}\frac{a^k b^k t^k\, k}{\Gamma(\alpha k+\beta+1)k!} d(t;q) \qquad (2.2)$$

NOW THE FOLLOWING OBSERVATION ARE MADE

(i) $\sum_{k=o}^{\infty}\frac{(a_1)_k \ \ldots\ldots (a_p)_k}{(b_1)_k \ \ldots\ldots (b_q)_k}\frac{a^k\, b^k t^k k}{\Gamma(\alpha k+\beta+1)k!}$ converges absolutely and therefore uniformly on domain of x over the region of integration.

(ii) $\int_0^1 \left|(1-tq)_{-\mu-1} t^{\lambda-1}\right| \, d(t;q)$ IS CONVERGENT,

PROVIDED RE $(\lambda) > 0$, RE $(\mu) < 0$, $0 < |q| < 1$

THEREFORE THE ORDER OF INTEGRATION AND SUMMATION CAN BE INTERCHANGED IN (2.2) TO OBTAIN.

$$= \frac{z^{\lambda-\mu-1}}{\mathbf{\Gamma}_{\mathrm{q}}(-\mu)}\sum_{k=o}^{\infty}\frac{(a_1)_k \ \ldots\ldots (a_p)_k}{(b_1)_k \ \ldots\ldots (b_q)_k}\frac{a^k\, b^k k}{\Gamma(\alpha k+\beta+1)k!}\int_0^1 (1-tq)_{-\mu-1}\, t^{\lambda+k-1} d(t;q)$$

$$= \sum_{k=o}^{\infty}\frac{(a_1)_k \ \ldots\ldots (a_p)_k}{(b_1)_k \ \ldots\ldots (b_q)_k}\frac{a^k\, b^k k}{\Gamma(\alpha k+\beta+1)k!} D_{z,q}^{\mu} z^{k+\lambda-1}$$

Hence the statement (2.1) is proved.

3. Some special cases:

(i) If we take $= 0, \beta = 0$, Constant k=1,a=b=1 in equation (2.1) it becomes the fractional q-derivative of power series.

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}\sum_{k=o}^{\infty}\frac{(a_1)_k \ \ldots\ldots (a_p)_k}{(b_1)_k \ \ldots\ldots (b_q)_k}\frac{1}{\Gamma(\alpha k+\beta+1)k!} z^k\right\}$$

$$= \sum_{k=o}^{\infty}\frac{(a_1)_k \ \ldots\ldots (a_p)_k}{(b_1)_k \ \ldots\ldots (b_q)_k}\frac{1}{\Gamma(\alpha k+\beta+1)k!} D_{z,q}^{\mu}\{z^{k+\lambda-1}\} \qquad (3.1)$$

This equation (3.1) is known result given by Yadav and Purohit [8] and Ali, Jain and Sharma [9].

(ii) When $\alpha = 1, \beta = 0, a = 1, b = 1$, Constant n=1 and no upper or lower parameter in(2.1), we have

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}\sum_{k=o}^{\infty}\frac{z^k}{\Gamma(k+1)}\right\}=\sum_{k=o}^{\infty}\frac{1}{k!}D_{z,q}^{\mu}\left\{z^{k+\lambda-1}\right\} \tag{3.2}$$

EQUIVALENTLY,

$$D_{z,q}^{\mu}\left\{z^{\lambda-1}e^{z}\right\}=\sum_{k=o}^{\infty}\frac{1}{k!}D_{z,q}^{\mu}\left\{z^{k+\lambda-1}\right\} \tag{3.3}$$

Thus the equation reduces to fractional q-derivative of exponential function.

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