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## RESEARCH ARTICLE

## Dirichlet Average of S- Function and Fractional Derivative

Manoj Sharma

Department of Mathematics RJIT, BSF Academy, Tekanpur.

**Abstract:**

In this paper we establish a relation Dirichlet average of S-function, using fractional derivative.

**Key Words:** Dirichlet average, S- function and Fractional calculus operators.

**Mathematics Subject Classification:** 26A33, 33A30, 33A25 and 83C99.

**1. Introduction:**

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions like  $x^t, e^x$  etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging  $x^n, e^x$  etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

In this paper the Dirichlet average of Hyper-geometric function has been obtained.

**2. Definitions:**

We give below some of the definitions which are necessary in the preparation of this paper.

**2.1 Standard Simplex in  $R^n, n \geq 1$ :**

We denote the standard simplex in  $R^n, n \geq 1$  by [1, p.62].

$$E = E_n = \{S(u_1, u_2, \dots, u_n) : u_1 \geq 0, \dots, u_n \geq 0, u_1 + u_2 + \dots + u_n \leq 1\} \quad (2.1.1)$$

**2.2 Dirichlet measure:**

Let  $b \in C^k, k \geq 2$  and let  $E = E_{k-1}$  be the standard simplex in  $R^{k-1}$ . The complex measure  $\mu_b$  is defined by E[1].

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots u_{k-1}^{b_{k-1}-1} (1 - u_1 - \dots - u_{k-1})^{b_k-1} du_1 \dots du_{k-1} \quad (2.2.1)$$

Will be called a Dirichlet measure.

Here

$$B(b) = B(b_1, \dots, b_k) = \frac{\Gamma(b_1) \dots \Gamma(b_k)}{\Gamma(b_1 + \dots + b_k)},$$

$$C_{>} = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \pi/2\},$$

Open right half plane and  $C_{>}^k$  is the  $k^{\text{th}}$  Cartesian power of  $C_{>}$

**2.3 Dirichlet Average[1, p.75]:**

Let  $\Omega$  be the convex set in  $C_{>}^k$ , let  $z = (z_1, \dots, z_k) \in \Omega^k, k \geq 2$  and let  $u.z$  be a convex combination of  $z_1, \dots, z_k$ . Let  $f$  be a measurable function on  $\Omega$  and let  $\mu_b$  be a Dirichlet measure on the standard simplex  $E$  in  $R^{k-1}$ . Define

$$F(b, z) = \int_E f(u, z) d\mu_b(u) \tag{2.3.1}$$

We shall call F the Dirichlet measure of  $f$  with variables  $z = (z_1, \dots, z_k)$  and parameters  $b = (b_1, \dots, b_k)$ . Here

$$u, z = \sum_{i=1}^k u_i z_i \text{ and } u_k = 1 - u_1 - \dots - u_{k-1} \tag{2.3.2}$$

If  $k = 1$ , define  $F(b, z) = f(z)$ .

**2.4 Fractional Derivative [8, p.181]:**

The concept of fractional derivative with respect to an arbitrary function has been used by Erdelyi[8]. The most common definition for the fractional derivative of order  $\alpha$  found in the literature on the ‘‘Riemann-Liouville integral’’ is

$$D_z^\alpha F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z - t)^{-\alpha-1} dt \tag{2.4.1}$$

Where  $Re(\alpha) < 0$  and  $F(x)$  is the form of  $x^p f(x)$ , where  $f(x)$  is analytic at  $x = 0$ .

**3.1 S-Function:**

This function introduced by the author is defined as follows:

$${}^a S_{p,q}^{\alpha,\beta} (a_1 \dots a_p; b_1 \dots b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} k \frac{a^n x^n}{\Gamma(\alpha n + \beta + 1)n!} \tag{3.1}$$

Here,  $p$  upper parameters  $a_1, a_2, \dots, a_p$  and  $q$  lower parameters  $b_1, b_2, \dots, b_q$ ,  $\alpha, \beta \in C$ ,  $R(\alpha) > 0$ ,  $R(\beta) > 0$  and  $(a_j)_k (b_j)_k$  are pochhammer symbols and  $K$  is constant. The function (3) is defined when none of the denominator parameters  $b_j$ s,  $j = 1, 2, \dots, q$  is a negative integer or zero. If any parameter  $a_j$  is negative then the function (3) terminates into a polynomial in  $x$ . By using ratio test, it is evident that function (3) is convergent for all  $x$ , when  $q \geq p$ , it is convergent for  $|x| < 1$  when  $p = q + 1$ , divergent when  $p > q + 1$ . In some cases the series is convergent for  $x = 1$ ,  $x = -1$ . Let us consider take,

$$\beta = \sum_{j=1}^p a_j - \sum_{j=1}^q b_j$$

when  $p = q + 1$ , the series is absolutely convergent for  $|x| = 1$  if  $R(\beta) < 0$ , convergent for  $x = -1$ , if  $0 \leq R(\beta) < 1$  and divergent for  $|x| = 1$ , if  $1 \leq R(\beta)$ .

**3.2 Equivalence:**

In this section we shall show the equivalence of single Dirichlet average of  ${}^a S_{p,q}^{\alpha,\beta} (x)$  function ( $k = 2$ ) with the fractional derivative i.e.

$$S(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma\beta} (x - y)^{1-\beta-\beta'} D_{x-y}^{-\beta'} {}^a S_{p,q}^{\alpha,\beta} (x) (x - y)^{\beta-1} \tag{3.2}$$

**Proof:**

$$\begin{aligned} S(\beta, \beta'; x, y) &= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} k \frac{a^n x^n}{\Gamma(\alpha n + \beta + 1)} R_n(\beta, \beta'; x, y) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} k \frac{a^n}{\Gamma(\alpha n + \beta + 1)} \frac{\Gamma(\beta + \beta')}{\Gamma\beta \Gamma\beta'} \int_0^1 [ux + (1 - u)y]^n u^{\beta-1} (1 - u)^{\beta'-1} du \end{aligned}$$

Putting  $u(x - y) = t$ , we have,

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} k \frac{a^n x^n}{\Gamma(\alpha n + \beta + 1)} \frac{\Gamma(\beta + \beta')}{\Gamma\beta \Gamma\beta'} \int_0^{x-y} [t+y]^n \left(\frac{t}{x-y}\right)^{\beta-1} \left(1 - \frac{t}{x-y}\right)^{\beta'-1} \frac{dt}{x-y}$$

On changing the order of integration and summation, we have

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta + \beta')}{\Gamma\beta \Gamma\beta'} \int_0^{x-y} \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} k \frac{a^n}{\Gamma(\alpha n + \beta + 1)} [t+y]^n (t)^{\beta-1} (x-y-t)^{\beta'-1} dt$$

Or

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta + \beta')}{\Gamma\beta \Gamma\beta'} \int_0^{x-y} {}_a S_{p,q}^{\alpha,\beta} (t)^{\beta-1} (x-y-t)^{\beta'-1} dt$$

Hence by the definition of fractional derivative, we get

$$S(\beta, \beta'; x, y) = (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta + \beta')}{\Gamma\beta} D_{x-y}^{-\beta'} {}_a S_{p,q}^{\alpha,\beta} (x) (x-y)^{\beta-1}$$

This completes the Analysis.

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#### References

- [1] Carlson, B.C., Special Function of Applied Mathematics, Academic Press, New York, 1977.
- [2] Carlson, B.C., Appell's function F4 as a double average, SIAM J.Math. Anal.6 (1975), 960-965.
- [3] Carlson, B.C., Hidden symmetries of special functions, SIAM Rev. 12 (1970), 332-345.
- [4] Carlson, B.C., Dirichlet averages of  $x \log x$ , SIAM J.Math. Anal. 18(2) (1987), 550-565.
- [5] Carlson, B.C., A connection between elementary functions and higher transcendental functions, SIAM J. Appl. Math. 17 (1969), 116-140.
- [6] Deora, Y. and Banerji, P.K., Double Dirichlet average of  $e^x$  using fractional derivatives, J.Fractional Calculus 3 (1993), 81-86.
- [7] Deora, Y. and Banerji, P.K., Double Dirichlet average of  $x^t$  and fractional derivatives, Rev.Tec.Ing.Univ. Zulia 16(2) (1993), 157-161.
- [8] Deora, Y, and Banerji, P,K, An Application of Fractional Calculus to the solution of Euler-Darbox equation in terma of Dirichlet average J. of fractional Calculus vol.5, may (1994) 91-94.
- [9] Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., Tables of Integral Transforms, Vol.2 McGraw-Hill, New York, 1954.
- [10] Gupta,S.C. and Agrawal, B.M., Dirichlet average and fractional derivatives, J. Indian Acad.Math. 12(1) (1990), 103-115.
- [11] Gupta,S.C. and Agrawal, Double Dirichlet average of  $e^x$  using fractional derivatives, Ganita Sandesh 5 (1) (1991),47-52.
- [12] Kilbas, A., A. and Kattuveetti, Anitha,,: representations of dirichlet averages of generalized mittag-leffler function viafractional integrals and special functions, FCAA Vol.11(4) (2008) 471-492.
- [13] Mathai, A.M. and Saxena,R.K., The H-Function with Applications in Statistics and other Disciplines, Wiley Halsted, New York, 1978.
- [14] Saxena,R.K., Mathai,A.M and Haubold, H.J., Unified fractional kinetic equation and a fractional diffusion equation, J. Astrophysics and Space Science 209 (2004) , 299-310.
- [15] Saxena, R. K., Pogany, T. K., Ram, J. and Daiya, J.: Dirichlet Averages of Generalized Multi-index Mittag-Leffler Functions, AJM, Vol. 3, No. 4, (2010), 174-187.
- [16] Sharma, Manoj and Jain, Renu,,: Triple Dirichlet average of  $e^x$  and fractional derivative, Applied Science Periodicals, (2008) 163-168
- [17] Sharma, M., Jain, R., Sharma, K.: Dirichlet average of M-L Function and Fractional Derivative, Napier Indian Advanced Research Journal of Sciences, (2010) 05-06.

- [18] Sharma, Manoj and Jain, Renu,: Double Dirichlet average of  $\cos x$  and Fractional Derivative, Ganita Sandesh, India, (2007) 107-110.
- [19] Sharma, Manoj and Jain, Renu,: Double Dirichlet average of  $x^t \log x$  and Fractional Derivative, Journal of Indian Acad. Math, India, (2006) 337-342.
- [20] Sharma, Manoj and Jain, Renu, Dirichlet Average of  $\cosh x$  and Fractional Derivative, South East Asian J. of Math. and Math. Sci. (2007).
- [21] Sharma, M.: Fractional Integration and Fractional Differentiation of the M-Series, Fract. Calc. Appl. Anal. 12 No.4 (2008) 187-191.
- [22] Sharma, M. and Jain, R. : A note on a generalized M-series as a special function of fractional calculus, Fract. Calc. Appl. Anal. 12 No.4 (2009) 449-452.
- [23] Sharma, K. and Dhakar, V. S.:On fractional calculus and certain results involving  $K_2$  function,G.J.S.F.R, Vol. 11(5) 2011.