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ITERATIVE I-CIRCUMCEVIAN TRIANGLES

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Abstract

As in two other papers regarding I-cevian and I-circumcevian triangles, respectively, in this paper we propose, using logical deductibility relations and the analogy method, to present some interesting results in Triangle Geometry, more specifically regarding iterative I-circumcevian triangles. Thus, we consider a triangle ABC and the interior bisectors of the angles of the triangle, which intersect at point I and which intersect the sides of the given triangle at points A′, B′ and C′, and the circumcircle of triangle ABC at A_1, B_1 and C_1. Then, we will call the triangle A′B′C′ the I-cevian triangle attached to the triangle ABC and the point I, and we will call the triangle $A_1B_1C_1$ the I-circumcevian triangle attached to the triangle ABC and the point I. In the following, for any $n\in\mathbf{N}$, $n\geq2$, we will denote by $A_nB_nC_n$ the I-circumcevian triangle attached to the $A_{n-1}B_{n-1}C_{n-1}$ triangle. Using common mathematical knowledge, valid in any triangle, but also the results presented in the two papers mentioned above, we can obtain a series of very interesting geometric or trigonometric identities and inequalities, some of them very difficult to prove synthetically. The work is exclusively about Mathematics Didactics and is addressed to both students and teachers eager for performance in this field of Mathematics or, in Mathematics in general.

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Introduction: -

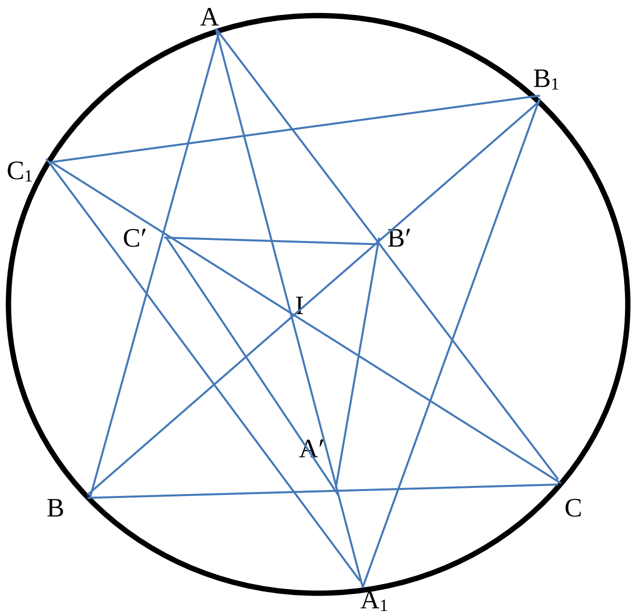
Posing the problem

This paper is a particularization and continuation of the paper (Vălcan, 2022), which, in turn, is a particularization and continuation of the paper (Vălcan, 2021).

Thus, in this paragraph, we will present the main results obtained in (Vălcan, 2022), starting from the results in (Vălcan, 2021). In this sense, we consider a triangle ABC and the following definitions – see the figure below.

Continuing the general ideas from (Vălcan, 2021) for any three cevians and particularizing them for the case of bisectors, in (Vălcan, 2022) we referred to the I-cevian triangle and the I-circumcevian triangle, attached to a triangle ABC and point I – the center of the circle inscribed in this triangle ABC.

Thus, in (Vălcan, 2022), considering the figure below, in which [AA′, [BB′and[CC′ are the bisectors of angles A, B and C of triangle ABC, with A′∈(BC), B′∈(CA) and C′∈(AB), which intersect at point I – the circle of the circle inscribed in triangle ABC and in which A_1, B_1, C_1 are the points where these internal bisectors intersect the circumcircle of the triangle for the second time, triangle A′B′C′ we called the I-cevian triangle attached to triangle ABC and point I, and triangle $A_1B_1C_1$ we called the I-circumcevian triangle attached to triangle ABC and point I.



We specify that the triangle A′B′C′ is also called the pedal triangle associated with the triangle ABC.

In this paper we will refer to iterative I-circumcevian triangles, that is, to those triangles that are I-circumcevian for other I-circumcevian triangles.

As in the works mentioned above, we will denote by a, b, c the lengths of the sides of triangle ABC, by a′, b′ and c′ the lengths of the sides of triangle A′B′C′ and by a_1, b_1 and c_1 the lengths of the sides of triangle $A_1B_1C_1$. We will also denote by S′, p′ and r′ - the area, semiperimeter and radius of the circle inscribed in the triangle A′B′C′ and by S_1, p_1 and r_1 - the area, semiperimeter and radius of the circle inscribed in the triangle $A_1B_1C_1$.

We will keep the numbering from (Vălcan, 2022) and continue from there.

As we showed in (Vălcan, 2022), the following equalities hold:

1) $\sphericalangle B_1A_1C_1 \overset{not.}{=} A_1=\frac{B+C}{2}$ and the analogues:

$\sphericalangle A_1B_1C_1 \overset{not.}{=} B_1=\frac{C+A}{2}$. and $\sphericalangle B_1C_1A_1 \overset{not.}{=} C_1=\frac{A+B}{2}$. (3.6)

or:

$\sphericalangle C_1A_1B_1 \overset{not.}{=} A_1=\frac{\pi-A}{2}$, $\sphericalangle A_1B_1C_1 \overset{not.}{=} B_1=\frac{\pi-B}{2}$,

and $\sphericalangle B_1C_1A_1 \overset{not.}{=} C_1=\frac{\pi-C}{2}$. (3.6′)

2) $a_1=2\cdot R\cdot \sin A_1=2\cdot R\cdot \sin\frac{B+C}{2}=2\cdot R\cdot \cos\frac{A}{2}$ and the analogues:

$b_1=2\cdot R\cdot \sin B_1=2\cdot R\cdot \sin\frac{A+C}{2}=2\cdot R\cdot \cos\frac{B}{2}$, $c_1=2\cdot R\cdot \sin C_1=2\cdot R\cdot \sin\frac{A+B}{2}=2\cdot R\cdot \cos\frac{C}{2}$. (3.7)

3) $p_1=\frac{a_1+b_1+c_1}{2}=R\cdot\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$; (3.8) or:

$$p_1=\frac{2\cdot R_1\cdot \sin A_1+2\cdot R_1\cdot \sin B_1+2\cdot R_1\cdot \sin C_1}{2}$$

$$=R\cdot(\sin A_1+\sin B_1+\sin C_1)=4\cdot R\cdot\cos\frac{A_1}{2}\cdot\cos\frac{B_1}{2}\cdot\cos\frac{C_1}{2}$$

$$=4\cdot R\cdot\cos\frac{A+B}{4}\cdot\cos\frac{B+C}{4}\cdot\cos\frac{A+C}{4}$$

$$=4\cdot R\cdot\cos\frac{\pi-A}{4}\cdot\cos\frac{\pi-B}{4}\cdot\cos\frac{\pi-C}{4}. \qquad (3.8')$$

4) $$S_1=\frac{a_1\cdot b_1\cdot c_1}{4\cdot R}=2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1=2\cdot R^2\cdot\sin\frac{A+B}{2}\cdot\sin\frac{B+C}{2}\cdot\sin\frac{A+C}{2}$$

$$=2\cdot R^2\cdot\sin\frac{\pi-A}{2}\cdot\sin\frac{\pi-B}{2}\cdot\sin\frac{\pi-C}{2}=2\cdot R^2\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}, \qquad (3.9)$$

or:

$$S_1=2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1=2\cdot R^2\cdot\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\cdot\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)\cdot\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$=2\cdot R^2\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}. \qquad (3.9')$$

5) $$r_1=\frac{S_1}{p_1}=4\cdot R\cdot\sin\frac{A_1}{2}\cdot\sin\frac{B_1}{2}\cdot\sin\frac{C_1}{2}=4\cdot R\cdot\sin\frac{A+B}{4}\cdot\sin\frac{B+C}{4}\cdot\sin\frac{A+C}{4}$$

$$=4\cdot R\cdot\sin\frac{\pi-A}{4}\cdot\sin\frac{\pi-B}{4}\cdot\sin\frac{\pi-C}{4}. \qquad (3.10)$$

or:

$$r_1=\frac{S_1}{p_1}=\frac{2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1}{R\cdot\left(\cos\frac{A_0}{2}+\cos\frac{B_0}{2}+\cos\frac{B_0}{2}\right)}=2\cdot R\cdot\frac{\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}, \qquad (3.10')$$

or:

$$r_1=4\cdot R\cdot\sin\frac{A_1}{2}\cdot\sin\frac{B_1}{2}\cdot\sin\frac{C_1}{2}=4\cdot R\cdot\sin\frac{\pi-A}{4}\cdot\sin\frac{\pi-B}{4}\cdot\sin\frac{\pi-C}{4}. \qquad (3.10'')$$

Main Results:-

In this paragraph we will study iterative I-circumcised triangles. So now we begin an iterative process. Therefore, let be:

- $A_2B_2C_2$ the I_1-circumcevian triangle of the triangle $A_1B_1C_1$, that is the triangle formed by the intersection points of the bisectors $[A_1A_2$, $[B_1B_2$ and $[C_1C_2$ of the triangle $A_1B_1C_1$, with its circumscribed circle;
- $A_3B_3C_3$ the I_2-circumcevian triangle of the triangle $A_2B_2C_2$, that is the triangle formed by the intersection points of the bisectors $[A_2A_3$, $[B_2B_3$ and $[C_2C_3$ of the triangle $A_2B_2C_2$, with its circumscribed circle;
- $A_4B_4C_4$ the I_3-circumcevian triangle of the triangle $A_3B_3C_3$, that is the triangle formed by the intersection points of the bisectors $[A_3A_4$, $[B_3B_4$ and $[C_3C_4$ of the triangle $A_3B_3C_3$, with its circumscribed circle;
- ...
- $A_nB_nC_n$ the I_{n-1}-circumcevian triangle of the triangle $A_{n-1}B_{n-1}C_{n-1}$, that is the triangle formed by the intersection points of the bisectors $[A_{n-1}A_n$, $[B_{n-1}B_n$ and $[C_{n-1}C_n$ of the triangle $A_{n-1}B_{n-1}C_{n-1}$, with its circumscribed circle.

Then, according to equalities (3.6) and (3.6′), the following equalities hold:

$$A_1=\frac{B+C}{2}=\frac{\pi-A}{2},\qquad B_1=\frac{C+A}{2}=\frac{\pi-B}{2}\qquad\text{and}\qquad C_1=\frac{A+B}{2}=\frac{\pi-C}{2},$$

$$A_2=\frac{B_1+C_1}{2}=\frac{\pi+A}{4},\qquad B_2=\frac{C_1+A_1}{2}=\frac{\pi+B}{4}\qquad\text{and}\qquad C_2=\frac{A_1+B_1}{2}=\frac{\pi+C}{4},$$

$A_3=\frac{B_2+C_2}{2}=\frac{3\cdot\pi-A}{8}$, $B_3=\frac{C_2+A_2}{2}=\frac{3\cdot\pi-B}{8}$ and $C_3=\frac{A_2+B_2}{2}=\frac{3\cdot\pi-C}{8}$,

$A_4=\frac{B_2+C_2}{2}=\frac{5\cdot\pi+A}{16}$, $B_4=\frac{C_2+A_2}{2}=\frac{5\cdot\pi+B}{16}$ and $C_4=\frac{A_2+B_2}{2}=\frac{5\cdot\pi+C}{16}$,

$A_5=\frac{B_4+C_4}{2}=\frac{11\cdot\pi-A}{32}$, $B_5=\frac{C_4+A_4}{2}=\frac{11\cdot\pi-B}{32}$ and $C_5=\frac{A_4+B_4}{2}=\frac{11\cdot\pi-C}{32}$,

$A_6=\frac{B_5+C_5}{2}=\frac{21\cdot\pi+A}{64}$, $B_6=\frac{C_5+A_5}{2}=\frac{21\cdot\pi+B}{64}$ and $C_6=\frac{A_5+B_5}{2}=\frac{21\cdot\pi+C}{64}$,

...

$A_n=\frac{B_{n-1}+C_{n-1}}{2}$, $B_n=\frac{C_{n-1}+A_{n-1}}{2}$ and $C_n=\frac{A_{n-1}+B_{n-1}}{2}$. (4.1)

To determine the angles A_n, B_n and C_n we proceed as follows: we consider three numbers x, y, $z\in(0,+\infty)$ such that:

$x+y+z=k$ (4.2)

and also, we consider three sequences x_n, y_n and z_n defined as follows:

$x_0=x$, $y_0=y$ and $z_0=z$, (4.3)

and, for every $n\in\mathbf{N}$,

$x_n=\frac{y_{n-1}+z_{n-1}}{2}$, $y_n=\frac{z_{n-1}+x_{n-1}}{2}$ and $z_n=\frac{x_{n-1}+y_{n-1}}{2}$. (4.4)

Then:

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}=\frac{1}{2}\cdot\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix}. \quad (4.5)$$

If we note:

$$\Delta_n=\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} \text{ and } \Omega=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad (4.6)$$

then the equality (4.6) above becomes:

$\Delta_n=\Omega\cdot\Delta_{n-1}$. (4.7)

From equality (4.7), it follows that:

$$\Delta_n=\frac{1}{2}\cdot\Omega\cdot\Delta_{n-1}=\frac{1}{2^2}\cdot\Omega^2\cdot\Delta_{n-2}=\frac{1}{2^3}\cdot\Omega^3\cdot\Delta_{n-3}=\ldots=\frac{1}{2^{n-1}}\cdot\Omega^{n-1}\cdot\Delta_1=\frac{1}{2^n}\cdot\Omega^n\cdot\Delta_0. \quad (4.8)$$

On the other hand, from equality (4.5), it follows that:

$$\Omega=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad (4.9)$$

and, thus:

$$\Omega^2=\Omega\cdot\Omega=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

$$\Omega^3=\Omega^2\cdot\Omega=\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix},$$

$$\Omega^4=\Omega^3\cdot\Omega=\begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{pmatrix},$$

$$\Omega^5=\Omega^4\cdot\Omega=\begin{pmatrix}6&5&5\\5&6&5\\5&5&6\end{pmatrix}\cdot\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}=\begin{pmatrix}10&11&11\\11&10&11\\11&11&10\end{pmatrix},$$

$$\Omega^6=\Omega^4\cdot\Omega=\begin{pmatrix}10&11&11\\11&10&11\\11&11&10\end{pmatrix}\cdot\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}=\begin{pmatrix}22&21&21\\21&22&21\\21&21&22\end{pmatrix}.$$

We suppose that:

$$\Omega^n=\begin{pmatrix}\alpha_n&\beta_n&\beta_n\\\beta_n&\alpha_n&\beta_n\\\beta_n&\beta_n&\alpha_n\end{pmatrix}. \quad (4.10)$$

Then:

$$\Omega^{n+1}=\Omega^n\cdot\Omega=\begin{pmatrix}\alpha_n&\beta_n&\beta_n\\\beta_n&\alpha_n&\beta_n\\\beta_n&\beta_n&\alpha_n\end{pmatrix}\cdot\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

$$=\begin{pmatrix}2\cdot\beta_n&\alpha_n+\beta_n&\alpha_n+\beta_n\\\alpha_n+\beta_n&2\cdot\beta_n&\alpha_n+\beta_n\\\alpha_n+\beta_n&\alpha_n+\beta_n&2\cdot\beta_n\end{pmatrix}=\begin{pmatrix}\alpha_{n+1}&\beta_{n+1}&\beta_{n+1}\\\beta_{n+1}&\alpha_{n+1}&\beta_{n+1}\\\beta_{n+1}&\beta_{n+1}&\alpha_{n+1}\end{pmatrix}. \quad (4.11)$$

Therefore, from equalities (4.11), we obtain that:

$\alpha_{n+1}=2\cdot\beta_n$ and $\beta_{n+1}=\alpha_n+\beta_n$. (4.12)

Now, from equalities (4.12), it follows that:

$\alpha_{n+1}=2\cdot\beta_n=2\cdot(\alpha_{n-1}+\beta_{n-1})=2\cdot\alpha_{n-1}+2\cdot\beta_{n-1}=2\cdot\alpha_{n-1}+\alpha_n$; (4.13)

that is:

$\alpha_{n+1}-\alpha_n-2\cdot\alpha_{n-1}=0$. (4.13′)

The characteristic equation of this recurrence relation, (4.13′), is:

$r^2-r-2=0$, (4.14)

with the roots / solutions:

$r_1=-1$ and $r_2=2$. (4.14′)

From the equalities (4.14′), it follows that, for any $n\in\mathbf{N}$,

$\alpha_n=p\cdot(-1)^n+q\cdot2^n$. (4.15)

From equality (4.15), for:

$n=1$ and $n=2$

we obtain:

$-p+2\cdot q=0$ and $p+4\cdot q=2$,

from which it follows that:

$p=\frac{2}{3}$ and $q=\frac{1}{3}$. (4.16)

Therefore, from equalities (4.15), (4.16) and (4.13), it follows that:

$\alpha_n=\frac{2\cdot(-1)^n+2^n}{3}$ and $\beta_n=\frac{(-1)^{n+1}+2^n}{3}$. (4.17)

Now, from equalities (4.10) and (4.17), it follows that:

$$\Omega^n=\begin{pmatrix}\frac{2\cdot(-1)^n+2^n}{3}&\frac{(-1)^{n+1}+2^n}{3}&\frac{(-1)^{n+1}+2^n}{3}\\\frac{(-1)^{n+1}+2^n}{3}&\frac{2\cdot(-1)^n+2^n}{3}&\frac{(-1)^{n+1}+2^n}{3}\\\frac{(-1)^{n+1}+2^n}{3}&\frac{(-1)^{n+1}+2^n}{3}&\frac{2\cdot(-1)^n+2^n}{3}\end{pmatrix} \quad (4.18)$$

and, now, according to equality (4.8),

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \frac{1}{2^n} \cdot \begin{pmatrix} \frac{2\cdot(-1)^n+2^n}{3} & \frac{(-1)^{n+1}+2^n}{3} & \frac{(-1)^{n+1}+2^n}{3} \\ \frac{(-1)^{n+1}+2^n}{3} & \frac{2\cdot(-1)^n+2^n}{3} & \frac{(-1)^{n+1}+2^n}{3} \\ \frac{(-1)^{n+1}+2^n}{3} & \frac{(-1)^{n+1}+2^n}{3} & \frac{2\cdot(-1)^n+2^n}{3} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \tag{4.19}$$

that is:

$$x_n = \frac{2\cdot(-1)^n+2^n}{3\cdot 2^n}\cdot x + \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot y + \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot z = \frac{2\cdot(-1)^n+2^n}{3\cdot 2^n}\cdot x + \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot (y+z)$$

$$= \frac{2\cdot(-1)^n+2^n}{3\cdot 2^n}\cdot x + \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot (k-x) = \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot k + \frac{(-1)^n}{2^n}\cdot x. \tag{4.19′}$$

Analogously we obtain that:

$$y_n = \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot k + \frac{(-1)^n}{2^n}\cdot y \quad \text{and} \quad z_n = \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot k + \frac{(-1)^n}{2^n}\cdot z. \tag{4.19′}$$

Now, we observe that, for every $n\in\mathbf{N}$,

$$x_n+y_n+z_n = 3\cdot\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot k + \frac{(-1)^n}{2^n}\cdot(x+y+z) = \frac{(-1)^{n+1}+2^n}{2^n}\cdot k + \frac{(-1)^n}{2^n}\cdot k = k, \tag{4.2′}$$

Finally, for:

$k=\pi$, $x=A$, $y=B$ and $z=C$,

from the above equalities, we obtain all angles A_n, B_n and C_n.Thus, for every $n\in\mathbf{N}$,

$$A_n = \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi + \frac{(-1)^n}{2^n}\cdot A, \quad B_n = \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi + \frac{(-1)^n}{2^n}\cdot B$$

and

$$C_n = \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi + \frac{(-1)^n}{2^n}\cdot C. \tag{4.20}$$

Since, according to equalities (4.20),

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi + \frac{(-1)^n}{2^n}\cdot A\right] = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3},$$

it follows that this iterative process can be repeated infinitely, and the "limit" triangle is an equilateral one. So, starting from any triangle ABC, the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, ..., $A_nB_nC_n$ can be formed, for every $n\in\mathbf{N}^*$ and, according to equalities (4.20), the following equalities hold:

$A_1 = \frac{B+C}{2} = \frac{\pi-A}{2}$, $B_1 = \frac{C+A}{2} = \frac{\pi-B}{2}$ and $C_1 = \frac{A+B}{2} = \frac{\pi-C}{2}$,

$A_2 = \frac{B_1+C_1}{2} = \frac{\pi+A}{4}$, $B_2 = \frac{C_1+A_1}{2} = \frac{\pi+B}{4}$ and $C_2 = \frac{A_1+B_1}{2} = \frac{\pi+C}{4}$,

$A_3 = \frac{B_2+C_2}{2} = \frac{3\cdot\pi-A}{8}$, $B_3 = \frac{C_2+A_2}{2} = \frac{3\cdot\pi-B}{8}$ and $C_3 = \frac{A_2+B_2}{2} = \frac{3\cdot\pi-C}{8}$,

$A_4 = \frac{B_2+C_2}{2} = \frac{5\cdot\pi+A}{16}$, $B_4 = \frac{C_2+A_2}{2} = \frac{5\cdot\pi+B}{16}$ and $C_4 = \frac{A_2+B_2}{2} = \frac{5\cdot\pi+C}{16}$,

$A_5 = \frac{B_4+C_4}{2} = \frac{11\cdot\pi-A}{32}$, $B_5 = \frac{C_4+A_4}{2} = \frac{11\cdot\pi-B}{32}$ and $C_5 = \frac{A_4+B_4}{2} = \frac{11\cdot\pi-C}{32}$,

$A_6 = \frac{B_5+C_5}{2} = \frac{21\cdot\pi+A}{64}$, $B_6 = \frac{C_5+A_5}{2} = \frac{21\cdot\pi+B}{64}$ and $C_6 = \frac{A_5+B_5}{2} = \frac{21\cdot\pi+C}{64}$,

...

$$A_n = \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi + \frac{(-1)^n}{2^n}\cdot A, \quad B_n = \frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi + \frac{(-1)^n}{2^n}\cdot B$$

and $$C_n=\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi+\frac{(-1)^n}{2^n}\cdot C. \qquad (4.20')$$

We observe that, for a $n\in\mathbf{N}$,

$A_n=B_n$ if and only if $A=B$; (4.21)

in which case, for every $m\in\mathbf{N}$,

$A_m=B_m$. (4.22)

We deduce from this that, for an $n\in\mathbf{N}$, the triangle $A_nB_nC_n$ is isosceles / equilateral exactly if, for each $m\in\mathbf{N}$, the triangle $A_mB_mC_m$ is isosceles / equilateral. In particular, triangle ABC is isosceles / equilateral exactly if, for each $m\in\mathbf{N}$, triangle $A_mB_mC_m$ is isosceles / equilateral.

On the other hand, since, according to equalities (4.1):

$$A_n=\frac{B_{n-1}+C_{n-1}}{2}=\frac{\pi}{2}-\frac{A_{n-1}}{2}, \qquad B_n=\frac{C_{n-1}+A_{n-1}}{2}=\frac{\pi}{2}-\frac{B_{n-1}}{2},$$

and $$C_n=\frac{A_{n-1}+B_{n-1}}{2}\ \frac{\pi}{2}-\frac{C_{n-1}}{2},$$

it follows that, regardless of the measures of the angles of triangle ABC, for any $n\in\mathbf{N}^*$, triangle $A_nB_nC_n$ is (always!) acute-angled.

Further, for each $k\in\mathbf{N}^*$, we will determine the lengths of the sides a_k, b_k and c_k, the semiperimeter p_k, the area S_k and the radius of the circle inscribed in the triangle $A_kB_kC_k$.

Because:

$R_1=R$, (4.23)

from the Law of Sine, applied to triangle $A_1B_1C_1$, we obtain that:

$$\frac{B_1C_1}{\sin A_1}=2\cdot R. \qquad (4.24)$$

So, according to equalities (4.23), (4.24) and (4.1), we obtain that:

$$a_1=B_1C_1=2\cdot R\cdot\sin A_1=2\cdot R\cdot\sin\frac{B+C}{2}=2\cdot R\cdot\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)=2\cdot R\cdot\cos\frac{A}{2}. \qquad (4.25)$$

Analogously, we obtain that:

$$b_1=C_1A_1=2\cdot R\cdot\cos\frac{B}{2} \quad\text{and}\quad c_1=A_1C_1=2\cdot R\cdot\cos\frac{C}{2}. \qquad (4.25')$$

Thus, we obtained, again, the equalities from (3.7).

Judging analogously in the triangles $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$, $A_5B_5C_5$, $A_6B_6C_6$., ..., $A_nB_nC_n$, we obtain the equalities:

$$a_2=B_2C_2=2\cdot R\cdot\sin A_2=2\cdot R\cdot\sin\frac{B_1+C_1}{2}=2\cdot R\cdot\sin\left(\frac{\pi}{2}-\frac{A_1}{2}\right)=2\cdot R\cdot\cos\frac{A_1}{2}=2\cdot R\cdot\cos\frac{\pi-A}{4},$$

$$b_2=C_2A_2=2\cdot R\cdot\cos\frac{\pi-B}{4} \quad\text{and}\quad c_2=A_2C_2=2\cdot R\cdot\cos\frac{\pi-C}{4};$$

$$a_3=B_3C_3=2\cdot R\cdot\sin A_3=2\cdot R\cdot\sin\frac{B_2+C_2}{2}=2\cdot R\cdot\sin\left(\frac{\pi}{2}-\frac{A_2}{2}\right)=2\cdot R\cdot\cos\frac{A_2}{2}=2\cdot R\cdot\cos\frac{\pi+A}{8},$$

$$b_3=C_3A_3=2\cdot R\cdot\cos\frac{\pi+B}{8} \quad\text{and}\quad c_3=A_3C_3=2\cdot R\cdot\cos\frac{\pi+C}{8};$$

$$a_4=B_4C_4=2\cdot R\cdot\sin A_4=2\cdot R\cdot\sin\frac{B_3+C_3}{2}=2\cdot R\cdot\sin\left(\frac{\pi}{2}-\frac{A_3}{2}\right)=2\cdot R\cdot\cos\frac{A_3}{2}=2\cdot R\cdot\cos\frac{3\cdot\pi-A}{16};$$

$$b_4=C_4A_4=2\cdot R\cdot\cos\frac{3\cdot\pi-B}{16} \quad\text{and}\quad c_4=A_4C_4=2\cdot R\cdot\cos\frac{3\cdot\pi-C}{16};$$

$$a_5=B_5C_5=2\cdot R\cdot\sin A_5=2\cdot R\cdot\sin\frac{B_4+C_4}{2}=2\cdot R\cdot\sin\left(\frac{\pi}{2}-\frac{A_4}{2}\right)=2\cdot R\cdot\cos\frac{A_4}{2}=2\cdot R\cdot\cos\frac{5\cdot\pi+A}{32},$$

$b_5=C_5A_5=2\cdot R\cdot\cos\dfrac{5\cdot\pi+B}{32}$ and $c_5=A_5C_5=2\cdot R\cdot\cos\dfrac{5\cdot\pi+C}{32}$;

$$a_6=B_6C_6=2\cdot R\cdot\sin A_6=2\cdot R\cdot\sin\frac{B_5+C_5}{2}=2\cdot R\cdot\sin\left(\frac{\pi}{2}-\frac{A_5}{2}\right)=2\cdot R\cdot\cos\frac{A_5}{2}=2\cdot R\cdot\cos\frac{11\cdot\pi-A}{32},$$

$b_6=C_6A_6=2\cdot R\cdot\cos\dfrac{11\cdot\pi-B}{32}$ and $c_6=A_6C_6=2\cdot R\cdot\cos\dfrac{11\cdot\pi-C}{32}$;

...

$$a_n=B_nC_n=2\cdot R\cdot\sin A_n=2\cdot R\cdot\sin\frac{B_{n-1}+C_{n-1}}{2}=2\cdot R\cdot\sin\left(\frac{\pi}{2}-\frac{A_{n-1}}{2}\right)=2\cdot R\cdot\cos\frac{A_{n-1}}{2}$$

$$=2\cdot R\cdot\cos\left[\frac{(-1)^n+2^{n-1}}{3\cdot 2^n}\cdot\pi+\frac{(-1)^{n-1}}{2^n}\cdot A\right],$$

$b_n=C_nA_n=2\cdot R\cdot\cos\left[\dfrac{(-1)^n+2^{n-1}}{3\cdot 2^n}\cdot\pi+\dfrac{(-1)^{n-1}}{2^n}\cdot B\right]$ and

$$c_n=A_nC_n=2\cdot R\cdot\cos\left[\frac{(-1)^n+2^{n-1}}{3\cdot 2^n}\cdot\pi+\frac{(-1)^{n-1}}{2^n}\cdot C\right]. \tag{4.26}$$

Because:

$$\lim_{n\to\infty}A_n=\lim_{n\to\infty}\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi+\frac{(-1)^n}{2^n}\cdot A\right]=\lim_{n\to\infty}B_n=\lim_{n\to\infty}C_n=\frac{\pi}{3}, \tag{4.27}$$

it results that:

$$\lim_{n\to\infty}a_n=2\cdot R\cdot\lim_{n\to\infty}\cos\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi+\frac{(-1)^n}{2^n}\cdot A\right]=2\cdot R\cdot\cos\frac{\pi}{6}=2\cdot R\cdot\frac{\sqrt{3}}{2}=R\cdot\sqrt{3}. \tag{4.28}$$

Analogously, we obtain that:

$\lim\limits_{n\to\infty}b_n=R\cdot\sqrt{3}$ and $\lim\limits_{n\to\infty}c_n=R\cdot\sqrt{3}$, (4.28′)

that is, the sides a_n, b_n and c_n, tend, when $n\to\infty$, to the sides of the equilateral triangle inscribed in a circle of radius R.

Now, determining the semiperimeter p_k of triangle $A_kB_kC_k$ is immediate:

$$p_1=\frac{a_1+b_1+c_1}{2}=R\cdot\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$$

or:

$$p_1=\frac{a_1+b_1+c_1}{2}=\frac{2\cdot R_1\cdot\sin A_1+2\cdot R_1\cdot\sin B_1+2\cdot R_1\cdot\sin C_1}{2}$$

$$=R\cdot(\sin A_1+\sin B_1+\sin C_1)=4\cdot R\cdot\cos\frac{A_1}{2}\cdot\cos\frac{B_1}{2}\cdot\cos\frac{C_1}{2}$$

$$=4\cdot R\cdot\cos\frac{\pi-A}{4}\cdot\cos\frac{\pi-B}{4}\cdot\cos\frac{\pi-C}{4};$$

(See the equalities in (3.8) and (3.8′), respectively.)

$$p_2=\frac{a_2+b_2+c_2}{2}=4\cdot R\cdot\cos\frac{A_2}{2}\cdot\cos\frac{B_2}{2}\cdot\cos\frac{C_2}{2}=4\cdot R\cdot\cos\frac{\pi+A}{8}\cdot\cos\frac{\pi+B}{8}\cdot\cos\frac{\pi+C}{8};$$

$$p_3=\frac{a_3+b_3+c_3}{2}=4\cdot R\cdot\cos\frac{A_3}{2}\cdot\cos\frac{B_3}{2}\cdot\cos\frac{C_3}{2}=4\cdot R\cdot\cos\frac{3\cdot\pi-A}{16}\cdot\cos\frac{3\cdot\pi-B}{16}\cdot\cos\frac{3\cdot\pi-C}{16};$$

$$p_4=\frac{a_4+b_4+c_4}{2}=4\cdot R\cdot\cos\frac{A_4}{2}\cdot\cos\frac{B_4}{2}\cdot\cos\frac{C_4}{2}=4\cdot R\cdot\cos\frac{5\cdot\pi+A}{32}\cdot\cos\frac{5\cdot\pi+B}{32}\cdot\cos\frac{5\cdot\pi+C}{32};$$

$$p_5=\frac{a_5+b_5+c_5}{2}=4\cdot R\cdot\cos\frac{A_5}{2}\cdot\cos\frac{B_5}{2}\cdot\cos\frac{C_5}{2}=4\cdot R\cdot\cos\frac{11\cdot\pi-A}{64}\cdot\cos\frac{11\cdot\pi-B}{64}\cdot\cos\frac{11\cdot\pi-C}{64};$$

$$p_6=\frac{a_6+b_6+c_6}{2}=4\cdot R\cdot\cos\frac{A_6}{2}\cdot\cos\frac{B_6}{2}\cdot\cos\frac{C_6}{2}=4\cdot R\cdot\cos\frac{21\cdot\pi+A}{128}\cdot\cos\frac{21\cdot\pi+B}{128}\cdot\cos\frac{21\cdot\pi+C}{128};$$

...

$$p_n=\frac{a_n+b_n+c_n}{2}=4\cdot R\cdot\cos\frac{A_n}{2}\cdot\cos\frac{B_n}{2}\cdot\cos\frac{C_n}{2}$$

$$=4\cdot R\cdot\cos\left[\pi\cdot\frac{(-1)^{n+1}+2^n}{3\cdot 2^{n+1}}+\frac{(-1)^n}{2^{n+1}}\cdot A\right]\cdot\cos\left[\pi\cdot\frac{(-1)^{n+1}+2^n}{3\cdot 2^{n+1}}+\frac{(-1)^n}{2^{n+1}}\cdot B\right]$$

$$\cdot\cos\left[\pi\cdot\frac{(-1)^{n+1}+2^n}{3\cdot 2^{n+1}}+\frac{(-1)^n}{2^{n+1}}\cdot C\right]. \quad (4.29)$$

According to the above equalities, regardless of the way of determining p_n, we have:

$$\lim_{n\to\infty} p_n=\frac{3\cdot\sqrt{3}}{2}\cdot R, \quad (4.30)$$

that is, the perimeter p_n tends, when $n\to\infty$, to the perimeter of the equilateral triangle inscribed in a circle of radius R.

To calculate the area S_k of the triangle $A_kB_kC_k$ we proceed as follows:

$$S_k=\frac{b_k\cdot c_k\cdot\sin A_k}{2}. \quad (4.31)$$

So,

$$S_1=\frac{b_1\cdot c_1\cdot\sin A_1}{2}=\frac{2\cdot R\cdot\cos\frac{B}{2}\cdot 2\cdot R\cdot\cos\frac{C}{2}\cdot\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)}{2}=2\cdot R^2\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2},$$

or:

$$S_1=2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1=2\cdot R^2\cdot\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\cdot\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)\cdot\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$=2\cdot R^2\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}.$$

(See the equalities in (3.9) and (3.9′), respectively.)

Analogously, we obtain that:

$$S_2=2\cdot R^2\cdot\sin A_2\cdot\sin B_2\cdot\sin C_2=2\cdot R^2\cdot\sin\frac{\pi+A}{4}\cdot\sin\frac{\pi+B}{4}\cdot\sin\frac{\pi+C}{4};$$

$$S_3=2\cdot R^2\cdot\sin A_3\cdot\sin B_3\cdot\sin C_3=2\cdot R^2\cdot\sin\frac{3\cdot\pi-A}{8}\cdot\sin\frac{3\cdot\pi-B}{8}\cdot\sin\frac{3\cdot\pi-C}{8};$$

$$S_4=2\cdot R^2\cdot\sin A_4\cdot\sin B_4\cdot\sin C_4=2\cdot R^2\cdot\sin\frac{5\cdot\pi+A}{16}\cdot\sin\frac{5\cdot\pi+B}{16}\cdot\sin\frac{5\cdot\pi+C}{16};$$

$$S_5=2\cdot R^2\cdot\sin A_5\cdot\sin B_5\cdot\sin C_5=2\cdot R^2\cdot\sin\frac{11\cdot\pi-A}{32}\cdot\sin\frac{11\cdot\pi-B}{32}\cdot\sin\frac{11\cdot\pi-C}{32};$$

$$S_6=2\cdot R^2\cdot\sin A_6\cdot\sin B_6\cdot\sin C_6=2\cdot R^2\cdot\sin\frac{21\cdot\pi+A}{64}\cdot\sin\frac{21\cdot\pi+B}{64}\cdot\sin\frac{21\cdot\pi+C}{64};$$

...

$$S_n=2\cdot R^2\cdot\sin A_n\cdot\sin B_n\cdot\sin C_n$$

$$=2\cdot R^2\cdot\sin\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi+\frac{(-1)^n}{2^n}\cdot A\right]\cdot\sin\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi+\frac{(-1)^n}{2^n}\cdot B\right]$$

$$\cdot\sin\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi+\frac{(-1)^n}{2^n}\cdot C\right]. \quad (4.32)$$

And for S_n we have the equality:

$$\lim_{n\to\infty} S_n = \frac{3\cdot\sqrt{3}}{4}\cdot R^2, \tag{4.33}$$

that is, the area S_n tends, when $n \to\infty$, to the area of the equilateral triangle inscribed in a circle of radius R.

Applying the formula:

$$r = \frac{S}{p} \tag{4.34}$$

in each triangle $A_kB_kC_k$, we obtain:

$$r_1 = \frac{S_1}{p_1} = \frac{2\cdot R^2\cdot\sin A_1\cdot\sin B_1\cdot\sin C_1}{R\cdot\left(\cos\frac{A_0}{2}+\cos\frac{B_0}{2}+\cos\frac{B_0}{2}\right)} = 2\cdot R\cdot\frac{\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}},$$

(See the equalities in (3.10), (3.10′) and (3.10″).)

$$r_2 = \frac{S_2}{p_2} = \frac{2\cdot R^2\cdot\sin A_2\cdot\sin B_2\cdot\sin C_2}{R\cdot\left(\cos\frac{A_1}{2}+\cos\frac{B_1}{2}+\cos\frac{B_1}{2}\right)} = 2\cdot R\cdot\frac{\cos\frac{A_1}{2}\cdot\cos\frac{B_1}{2}\cdot\cos\frac{C_1}{2}}{\cos\frac{A_1}{2}+\cos\frac{B_1}{2}+\cos\frac{C_1}{2}}$$

$$= 2\cdot R\cdot\frac{\cos\frac{\pi-A}{4}\cdot\cos\frac{\pi-B}{4}\cdot\cos\frac{\pi-C}{4}}{\cos\frac{\pi-A}{4}+\cos\frac{\pi-B}{4}+\cos\frac{\pi-C}{4}} = 2\cdot R\cdot\frac{\sin\frac{\pi+A}{4}\cdot\sin\frac{\pi+B}{4}\cdot\sin\frac{\pi+C}{4}}{\sin\frac{\pi+A}{4}+\sin\frac{\pi+B}{4}+\sin\frac{\pi+C}{4}},$$

$$r_3 = \frac{S_3}{p_3} = \frac{2\cdot R^2\cdot\sin A_3\cdot\sin B_3\cdot\sin C_3}{R\cdot\left(\cos\frac{A_2}{2}+\cos\frac{B_2}{2}+\cos\frac{B_2}{2}\right)} = 2\cdot R\cdot\frac{\cos\frac{A_2}{2}\cdot\cos\frac{B_2}{2}\cdot\cos\frac{C_2}{2}}{\cos\frac{A_2}{2}+\cos\frac{B_2}{2}+\cos\frac{C_2}{2}}$$

$$= 2\cdot R\cdot\frac{\cos\frac{\pi+A}{8}\cdot\cos\frac{\pi+B}{8}\cdot\cos\frac{\pi+C}{8}}{\cos\frac{\pi+A}{8}+\cos\frac{\pi+B}{8}+\cos\frac{\pi+C}{8}} = 2\cdot R\cdot\frac{\sin\frac{3\cdot\pi-A}{8}\cdot\sin\frac{3\cdot\pi-B}{8}\cdot\sin\frac{3\cdot\pi-C}{8}}{\sin\frac{3\cdot\pi-A}{8}+\sin\frac{3\cdot\pi-B}{8}+\sin\frac{3\cdot\pi-C}{8}},$$

$$r_4 = \frac{S_4}{p_4} = \frac{2\cdot R^2\cdot\sin A_4\cdot\sin B_4\cdot\sin C_4}{R\cdot\left(\cos\frac{A_3}{2}+\cos\frac{B_3}{2}+\cos\frac{B_3}{2}\right)} = 2\cdot R\cdot\frac{\cos\frac{A_3}{2}\cdot\cos\frac{B_3}{2}\cdot\cos\frac{C_3}{2}}{\cos\frac{A_3}{2}+\cos\frac{B_3}{2}+\cos\frac{C_3}{2}}$$

$$= 2\cdot R\cdot\frac{\cos\frac{3\cdot\pi-A}{16}\cdot\cos\frac{3\cdot\pi-B}{16}\cdot\cos\frac{3\cdot\pi-C}{16}}{\cos\frac{3\cdot\pi-A}{16}+\cos\frac{3\cdot\pi-B}{16}+\cos\frac{3\cdot\pi-C}{16}} = 2\cdot R\cdot\frac{\sin\frac{5\cdot\pi-A}{16}\cdot\sin\frac{5\cdot\pi-B}{16}\cdot\sin\frac{5\cdot\pi-C}{16}}{\sin\frac{5\cdot\pi-A}{16}+\sin\frac{5\cdot\pi-B}{16}+\sin\frac{5\cdot\pi-C}{16}},$$

$$r_5 = \frac{S_5}{p_5} = \frac{2\cdot R^2\cdot\sin A_5\cdot\sin B_5\cdot\sin C_5}{R\cdot\left(\cos\frac{A_4}{2}+\cos\frac{B_4}{2}+\cos\frac{B_4}{2}\right)} = 2\cdot R\cdot\frac{\cos\frac{A_4}{2}\cdot\cos\frac{B_4}{2}\cdot\cos\frac{C_4}{2}}{\cos\frac{A_4}{2}+\cos\frac{B_4}{2}+\cos\frac{C_4}{2}}$$

$$= 2\cdot R\cdot\frac{\cos\frac{5\cdot\pi+A}{32}\cdot\cos\frac{5\cdot\pi+B}{32}\cdot\cos\frac{5\cdot\pi+C}{32}}{\cos\frac{5\cdot\pi+A}{32}+\cos\frac{5\cdot\pi+B}{32}+\cos\frac{5\cdot\pi+C}{32}} = 2\cdot R\cdot\frac{\sin\frac{11\cdot\pi-A}{32}\cdot\sin\frac{11\cdot\pi-B}{32}\cdot\sin\frac{11\cdot\pi-C}{32}}{\sin\frac{11\cdot\pi-A}{32}+\sin\frac{11\cdot\pi-B}{32}+\sin\frac{11\cdot\pi-C}{32}},$$

$$r_6=\frac{S_6}{p_6}=\frac{2\cdot R^2\cdot \sin A_6\cdot \sin B_6\cdot \sin C_6}{R\cdot\left(\cos\frac{A_5}{2}+\cos\frac{B_5}{2}+\cos\frac{B_5}{2}\right)}=2\cdot R\cdot\frac{\cos\frac{A_5}{2}\cdot\cos\frac{B_5}{2}\cdot\cos\frac{C_5}{2}}{\cos\frac{A_5}{2}+\cos\frac{B_5}{2}+\cos\frac{C_5}{2}}$$

$$=2\cdot R\cdot\frac{\cos\frac{11\cdot\pi-A}{64}\cdot\cos\frac{11\cdot\pi-B}{64}\cdot\cos\frac{11\cdot\pi-C}{64}}{\cos\frac{11\cdot\pi-A}{64}+\cos\frac{11\cdot\pi-B}{64}+\cos\frac{11\cdot\pi-C}{64}}=2\cdot R\cdot\frac{\sin\frac{21\cdot\pi+A}{64}\cdot\sin\frac{21\cdot\pi+B}{64}\cdot\sin\frac{21\cdot\pi+C}{64}}{\sin\frac{21\cdot\pi+A}{64}+\sin\frac{21\cdot\pi+B}{64}+\sin\frac{21\cdot\pi+C}{64}},$$

...

$$r_n=\frac{S_n}{p_n}=\frac{2\cdot R^2\cdot \sin A_n\cdot \sin B_n\cdot \sin C_n}{R\cdot\left(\cos\frac{A_{n-1}}{2}+\cos\frac{B_{n-1}}{2}+\cos\frac{B_{n-1}}{2}\right)}=2\cdot R\cdot\frac{\cos\frac{A_{n-1}}{2}\cdot\cos\frac{B_{n-1}}{2}\cdot\cos\frac{C_{n-1}}{2}}{\cos\frac{A_{n-1}}{2}+\cos\frac{B_{n-1}}{2}+\cos\frac{C_{n-1}}{2}}$$

$$=2\cdot R\cdot\frac{\prod\cos\left[\frac{(-1)^n+2^{n-1}}{3\cdot 2^n}\cdot\pi+\frac{(-1)^{n-1}}{2^n}\cdot A\right]}{\sum\cos\left[\frac{(-1)^n+2^{n-1}}{3\cdot 2^n}\cdot\pi+\frac{(-1)^{n-1}}{2^n}\cdot A\right]}=2\cdot R\cdot\frac{\prod\sin\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi-\frac{(-1)^n}{2^n}\cdot A\right]}{\sum\sin\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi-\frac{(-1)^n}{2^n}\cdot A\right]}. \tag{4.35}$$

Or:

$$r=4\cdot R\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}. \tag{4.36}$$

So,

$$r_1=4\cdot R\cdot\sin\frac{A_1}{2}\cdot\sin\frac{B_1}{2}\cdot\sin\frac{C_1}{2}=4\cdot R\cdot\sin\frac{\pi-A}{4}\cdot\sin\frac{\pi-B}{4}\cdot\sin\frac{\pi-C}{4},$$

$$r_2=4\cdot R\cdot\sin\frac{A_2}{2}\cdot\sin\frac{B_2}{2}\cdot\sin\frac{C_2}{2}=4\cdot R\cdot\sin\frac{\pi+A}{8}\cdot\sin\frac{\pi+B}{8}\cdot\sin\frac{\pi+C}{8},$$

$$r_3=4\cdot R\cdot\sin\frac{A_3}{2}\cdot\sin\frac{B_3}{2}\cdot\sin\frac{C_3}{2}=4\cdot R\cdot\sin\frac{3\cdot\pi-A}{16}\cdot\sin\frac{3\cdot\pi-B}{16}\cdot\sin\frac{3\cdot\pi-C}{16},$$

$$r_4=4\cdot R\cdot\sin\frac{A_4}{2}\cdot\sin\frac{B_4}{2}\cdot\sin\frac{C_4}{2}=4\cdot R\cdot\sin\frac{5\cdot\pi+A}{32}\cdot\sin\frac{5\cdot\pi+B}{32}\cdot\sin\frac{5\cdot\pi+C}{32},$$

$$r_5=4\cdot R\cdot\sin\frac{A_5}{2}\cdot\sin\frac{B_5}{2}\cdot\sin\frac{C_5}{2}=4\cdot R\cdot\sin\frac{11\cdot\pi-A}{64}\cdot\sin\frac{11\cdot\pi-B}{64}\cdot\sin\frac{11\cdot\pi-C}{64},$$

$$r_6=4\cdot R\cdot\sin\frac{A_6}{2}\cdot\sin\frac{B_6}{2}\cdot\sin\frac{C_6}{2}=4\cdot R\cdot\sin\frac{21\cdot\pi+A}{128}\cdot\sin\frac{21\cdot\pi+B}{128}\cdot\sin\frac{21\cdot\pi+C}{128},$$

...

$$r_n=4\cdot R\cdot\sin\frac{A_n}{2}\cdot\sin\frac{B_n}{2}\cdot\sin\frac{C_n}{2}=4\cdot R\cdot\sin\left[\pi\cdot\frac{(-1)^{n+1}+2^n}{3\cdot 2^{n+1}}+\frac{(-1)^n}{2^{n+1}}\cdot A\right]$$
$$\cdot\sin\left[\pi\cdot\frac{(-1)^{n+1}+2^n}{3\cdot 2^{n+1}}+\frac{(-1)^n}{2^{n+1}}\cdot B\right]\cdot\sin\left[\pi\cdot\frac{(-1)^{n+1}+2^n}{3\cdot 2^{n+1}}+\frac{(-1)^n}{2^{n+1}}\cdot C\right]. \tag{4.37}$$

Regardless of the way of determining the radius rn, according to the equations (4.35) and (4.37), we obtain that:

$$\lim_{n\to\infty} r_n=\frac{R}{2}, \tag{4.38}$$

that is, the radius r_n tends, when $n\to\infty$, to the radius of the circle inscribed in the equilateral triangle inscribed in a circle of radius R.

On the other hand, from the above equalities, we obtain that:

$$\frac{\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}=\sin\frac{\pi-A}{4}\cdot\sin\frac{\pi-B}{4}\cdot\sin\frac{\pi-C}{4}=\cos\frac{\pi+A}{4}\cdot\cos\frac{\pi+B}{4}\cdot\cos\frac{\pi+C}{4},$$

$$\frac{\sin\frac{\pi+A}{4}\cdot\sin\frac{\pi+B}{4}\cdot\sin\frac{\pi+C}{4}}{\sin\frac{\pi+A}{4}+\sin\frac{\pi+B}{4}+\sin\frac{\pi+C}{4}}=\sin\frac{\pi+A}{8}\cdot\sin\frac{\pi+B}{8}\cdot\sin\frac{\pi+C}{8}=\cos\frac{3\cdot\pi-A}{8}\cdot\cos\frac{3\cdot\pi-B}{8}\cdot\cos\frac{3\cdot\pi-C}{8},$$

$$\frac{\sin\frac{3\cdot\pi-A}{8}\cdot\sin\frac{3\cdot\pi-B}{8}\cdot\sin\frac{3\cdot\pi-C}{8}}{\sin\frac{3\cdot\pi-A}{8}+\sin\frac{3\cdot\pi-B}{8}+\sin\frac{3\cdot\pi-C}{8}}=\sin\frac{3\cdot\pi-A}{16}\cdot\sin\frac{3\cdot\pi-B}{16}\cdot\sin\frac{3\cdot\pi-C}{16}$$

$$=\cos\frac{5\cdot\pi+A}{16}\cdot\cos\frac{5\cdot\pi+B}{16}\cdot\cos\frac{5\cdot\pi+C}{16},$$

$$\frac{\sin\frac{5\cdot\pi-A}{16}\cdot\sin\frac{5\cdot\pi-B}{16}\cdot\sin\frac{5\cdot\pi-C}{16}}{\sin\frac{5\cdot\pi-A}{16}+\sin\frac{5\cdot\pi-B}{16}+\sin\frac{5\cdot\pi-C}{16}}=\sin\frac{5\cdot\pi+A}{32}\cdot\sin\frac{5\cdot\pi+B}{32}\cdot\sin\frac{5\cdot\pi+C}{32}$$

$$=\cos\frac{11\cdot\pi-A}{32}\cdot\cos\frac{11\cdot\pi-B}{32}\cdot\cos\frac{11\cdot\pi-C}{32},$$

$$\frac{\sin\frac{11\cdot\pi-A}{32}\cdot\sin\frac{11\cdot\pi-B}{32}\cdot\sin\frac{11\cdot\pi-C}{32}}{\sin\frac{11\cdot\pi-A}{32}+\sin\frac{11\cdot\pi-B}{32}+\sin\frac{11\cdot\pi-C}{32}}=\sin\frac{11\cdot\pi-A}{64}\cdot\sin\frac{11\cdot\pi-B}{64}\cdot\sin\frac{11\cdot\pi-C}{64}$$

$$=\cos\frac{21\cdot\pi+A}{64}\cdot\cos\frac{21\cdot\pi+B}{64}\cdot\cos\frac{21\cdot\pi+C}{64},$$

$$\frac{\sin\frac{21\cdot\pi+A}{64}\cdot\sin\frac{21\cdot\pi+B}{64}\cdot\sin\frac{21\cdot\pi+C}{64}}{\sin\frac{21\cdot\pi+A}{64}+\sin\frac{21\cdot\pi+B}{64}+\sin\frac{21\cdot\pi+C}{64}}=\sin\frac{21\cdot\pi+A}{128}\cdot\sin\frac{21\cdot\pi+B}{128}\cdot\sin\frac{21\cdot\pi+C}{128}$$

$$=\cos\frac{43\cdot\pi-A}{128}\cdot\cos\frac{43\cdot\pi-B}{128}\cdot\cos\frac{43\cdot\pi-C}{128},$$

...

$$\frac{\prod\sin\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi-\frac{(-1)^n}{2^n}\cdot A\right]}{\sum\sin\left[\frac{(-1)^{n+1}+2^n}{3\cdot 2^n}\cdot\pi-\frac{(-1)^n}{2^n}\cdot A\right]}=\prod\sin\left[\pi\cdot\frac{(-1)^{n+1}+2^n}{3\cdot 2^{n+1}}+\frac{(-1)^n}{2^{n+1}}\cdot A\right]$$

$$=\prod\cos\left[\pi\cdot\frac{(-1)^n+2^{n+1}}{3\cdot 2^{n+1}}-\frac{(-1)^n}{2^{n+1}}\cdot A\right].$$

Therefore, the equalities in (4.34) can be expressed in these terms.

Conclusions, Additions and Recommendations:-

As I have stated in previous works on this topic, we can study, in a general way, using logical deductibility and the method of analogy, the Cevian and Circumcevian triangles. On the principles underlying these two methods and their application in teaching, see (Vălcan, 2013).

We have seen in these works how these geometric or trigonometric relations (identities or inequalities) are transposed into cases known (or less known) to students, more precisely in I-cevian or circumcevian triangles.

Well, moreover, we saw above how the measures of the angles and the lengths of the sides of the I-circumcevian triangles can be calculated. Determining these allowed us to determine the semi-perimeters and areas of these triangles, and then the radii of the circles inscribed in them.

Of great help in these determinations was the fact that the radius of the circumscribed circle in each I-circumcevian triangle, regardless of the iteration order, was the same.

Next we will see what order relationships exist between the quantities determined in the previous paragraph, depending on the iteration order n.

First, we observe that the sequence $(A_n)_{n\in\mathbf{N}}$ is alternating, in the sense that:

- if$A\leq\frac{\pi}{3}$, th en:

$$A_{2\cdot n-2}\leq A_{2\cdot n} \quad \text{and} \quad A_{2\cdot n-1}\geq A_{2\cdot n+1}, \tag{5.1}$$

and, for every $n\in\mathbf{N}$,

$$A_{2\cdot n}\leq\frac{\pi}{3} \quad \text{and} \quad A_{2\cdot n+1}\geq\frac{\pi}{3}, \tag{5.2}$$

and:

- if $A\geq\frac{\pi}{3}$, then:

$$A_{2\cdot n-2}\geq A_{2\cdot n} \quad \text{and} \quad A_{2\cdot n-1}\leq A_{2\cdot n+1}. \tag{5.3}$$

and, for every $n\in\mathbf{N}$,

$$A_{2\cdot n}\geq\frac{\pi}{3} \quad \text{and} \quad A_{2\cdot n+1}\leq\frac{\pi}{3}. \tag{5.4}$$

Therefore, from inequalities (5.1) – (5.4), it follows that:

- if $A\leq\frac{\pi}{3}$, then:

$$A\leq A_2\leq A_4\leq \ldots \leq A_{2\cdot n-2}\leq A_{2\cdot n}\leq \ldots \leq\frac{\pi}{3}\leq\ldots\leq A_{2\cdot n+1}\leq A_{2\cdot n-1}\leq \ldots \leq A_3\leq A_1\leq\frac{\pi}{2} \tag{5.5}$$

and:

- if $A\geq\frac{\pi}{3}$, then:

$$A_1\leq A_3\leq A_3\leq \ldots \leq A_{2\cdot n-1}\leq A_{2\cdot n+1}\leq \ldots \leq\frac{\pi}{3}\leq\ldots\leq A_{2\cdot n}\leq A_{2\cdot n-2}\leq \ldots \leq A_2\leq\frac{\pi}{2}. \tag{5.6}$$

Now, according to formulas (4.25) and inequalities (5.5) and (5.6), it follows that:

- if $A\leq\frac{\pi}{3}$, then:

$$a\leq a_2\leq a_4\leq \ldots \leq a_{2\cdot n-2}\leq a_{2\cdot n}\leq \ldots \leq R\cdot\sqrt{3}\leq\ldots\leq a_{2\cdot n+1}\leq a_{2\cdot n-1}\leq \ldots \leq a_3\leq a_1\leq 2\cdot R \tag{5.7}$$

and:

- if $A\geq\frac{\pi}{3}$, then:

$$a_1\leq a_3\leq a_3\leq \ldots \leq a_{2\cdot n-1}\leq a_{2\cdot n+1}\leq \ldots \leq R\cdot\sqrt{3}\leq \ldots \leq a_{2\cdot n}\leq a_{2\cdot n-2}\leq \ldots \leq a_2. \tag{5.8}$$

Analogously, we obtain the inequalities for the angles B_k and C_k, and for the sides b_k and c_k, respectively, for ecvery $k\in\mathbf{N}^*$.

At the end of this paragraph, respectively of this paper, we will present the order relations between the semi-perimeters p_n, the areas S_n and the rays r_n, that is, we will present the monotonicity of the respective sequences.

In this sense, to begin with, we need the following notion and the following technical results:

Definition 5.1: *If I⊆**R** is an interval, then we say that the function f : I → **R** is semiconcave, if, for every x, y∈I, the following inequality holds*:

$$f\left(\frac{x+y}{2}\right)\geq\frac{f(x)+f(y)}{2}. \tag{5.9}$$

Proposition 5.2: *If I⊆**R** is an interval, then a function f : I → **R** is semiconcave if and only if, for any x, y, z∈I, the following inequality holds*:

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\geq f(x)+f(y)+f(z). \tag{5.10}$$

Proof: Let I⊆**R** be an interval and f : I →**R**, a function for which inequality (5.9) holds. Then, by hypothesis, for any x, y, z∈I,

$$f\left(\frac{x+y}{2}\right)\geq\frac{f(x)+f(y)}{2}, \quad f\left(\frac{y+z}{2}\right)\geq\frac{f(y)+f(z)}{2} \quad \text{and} \quad f\left(\frac{z+x}{2}\right)\geq\frac{f(z)+f(x)}{2}.$$

Adding these inequalities, member by member, we obtain that, for any x, y, z∈I, inequality (5.10) holds.
Conversely, for any x, y, z∈I, we have inequality (5.10), then considering z=x, in this inequality, we obtain inequality (5.9).

Corollary 5.3: *If A, B, C are the measures of the angles of a triangle ABC, then the inequalities hold*:

a) $$\sin A+\sin B+\sin C\leq\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}; \tag{5.11}$$

b) $$\sin A\cdot\sin B\cdot\sin C\leq\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}. \tag{5.12}$$

Proof: a) We consider the function:

f : (0,π) →**R**, where, for every x∈(0,π), f(x)=sinx.

Then the function f is semiconcave, because, for every x, y∈(0,π),

$$\sin\frac{x+y}{2}\geq\frac{\sin x+\sin y}{2} \quad\Leftrightarrow\quad 1\geq\cos\frac{x-y}{2},$$

what is obvious. Therefore, according to inequality (5.10), for:

x=A, y=B and z=C,

we obtain that:

$$\sin A+\sin B+\sin C\leq\sin\frac{A+B}{2}+\sin\frac{B+C}{2}+\sin\frac{C+A}{2},$$

inequality which is equivalent to inequality (5.11), because:

$$\sin\frac{A+B}{2}=\cos\frac{C}{2}, \quad \sin\frac{B+C}{2}=\cos\frac{B}{2} \quad \text{and} \quad \sin\frac{C+A}{2}=\cos\frac{A}{2}. \tag{5.13}$$

b) We consider the function:

g : (0,π) →**R**, where, for every x∈(0,π), g(x)=lnsinx.

Then the function f is semiconcave, because, for every x, y∈(0,π),

$$\ln\sin\frac{x+y}{2}\geq\frac{\ln\sin x+\ln\sin y}{2} \quad\Leftrightarrow\quad \sin\frac{x+y}{2}\geq\sqrt{\sin x\cdot\sin y}$$

$$\Leftrightarrow\quad 2\cdot\sin^2\frac{x+y}{2}\geq\cos(x-y)-\cos(x+y)$$

$$\Leftrightarrow\quad 2\cdot\sin^2\frac{x+y}{2}\geq\cos(x-y)-\cos(x+y)$$

$$\Leftrightarrow\quad 1-\cos(x+y)\geq\cos(x-y)-\cos(x+y)$$

$$\Leftrightarrow\quad 1\geq\cos(x-y),$$

what is obvious. Therefore, according to inequality (5.10), for:

x=A, y=B și z=C,

we obtain that:

$$\ln\sin A+\ln\sin B+\ln\sin C\leq\ln\sin\frac{A+B}{2}+\ln\sin\frac{B+C}{2}+\ln\sin\frac{C+A}{2},$$

inequality that is equivalent to the inequality in the statement, due to equalities (5.13).

Remark 5.4: *From inequality (5.12), we deduce (immediately!) the well-known inequality*:

$$\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\le\frac{1}{8},$$

which is valid in any triangle ABC.

Now, let's get back to our problem. Thus, we have:

Theorem 5.5: *If in triangle ABC the bisectors of the angles* $\hat{A}$, $\hat{B}$, $\hat{C}$ *intersect the circumcircle of the triangle for the second time at points D, E, and F, respectively, then the inequalities hold*:

a) $P_{\Delta ABC}\le P_{\Delta DEF}$; (5.14)

b) $Area(\Delta ABC)\le Area(\Delta DEF)$. (5.15)

Here P_{XYZ} *we denote the perimeter of triangle XYZ.*

Proof: Consider the figure below. Then:

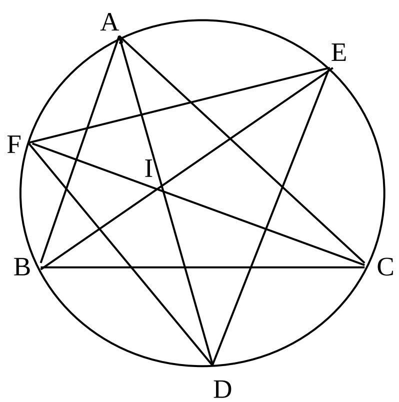
$$P_{\Delta ABC}=a+b+c=2\cdot R\cdot(\sin A+\sin B+\sin C), \tag{5.16}$$

and,

$$P_{\Delta DEF}=DE+EF+FD=2\cdot R\cdot[\sin(\sphericalangle DFE)+\sin(\sphericalangle EDF)+\sin((\sphericalangle DEF))]$$

$$=2\cdot R\cdot\left(\sin\frac{A+B}{2}+\sin\frac{B+C}{2}+\sin\frac{C+A}{2}\right)=2\cdot R\cdot\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right). \tag{5.17}$$

From equalities (5.16) and (5.17), it follows that the inequality in the statement is equivalent to the inequality (5.11).



b) Consider the figure above. Then:

$$A_{\Delta ABC}=\frac{a\cdot b\cdot c}{4\cdot R}=2\cdot R^2\cdot\sin A\cdot\sin B\cdot\sin C, \tag{5.18}$$

and,

$$A_{\Delta DEF}=\frac{DE\cdot EF\cdot FD}{4\cdot R}=2\cdot R^2\cdot\sin(\sphericalangle DFE)\cdot\sin(\sphericalangle EDF)\cdot\sin(\sphericalangle DEF)$$

$$=2\cdot R^2\cdot\sin\frac{A+B}{2}\cdot\sin\frac{B+C}{2}\cdot\sin\frac{C+A}{2}=2\cdot R^2\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}. \tag{5.19}$$

From equalities (5.18) and (5.18), it follows that the inequality in the statement is equivalent to inequality (5.12).

Returning to the I-circumcevian triangles, according to the results above, we obtain that:

$$p\le p_1\le p_2\le\ldots\le p_n\le\ldots\le\frac{3\cdot\sqrt{3}}{2}\cdot R \quad\text{and}\quad S\le S_1\le S_2\le\ldots\le S_n\le\ldots\le\frac{3\cdot\sqrt{3}}{4}\cdot R^2. \tag{5.20}$$

On the other hand, since the lnsinx function is semiconcave, according to inequality (5.10), it follows that:

$$\ln\sin\frac{A_n+B_n}{2}+\ln\sin\frac{B_n+C_n}{2}+\ln\sin\frac{C_n+A_n}{2}\ge\ln\sin A_n+\ln\sin B_n+\ln\sin C_n, \tag{5.21}$$

which is equivalent to:

$$\sin A_{n+1}\cdot\sin B_{n+1}\cdot\sin C_{n+1}\ge\sin A_n\cdot\sin B_n\cdot\sin C_n. \tag{5.21′}$$

It follows, from this and from equalities (4.36), that:

$$r\le r_1\le r_2\le\ldots\le r_n\le\ldots\le\frac{1}{2}\cdot R. \tag{5.22}$$

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