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RESEARCH ARTICLE

AN APPROACH FROM THE PERSPECTIVE TRIPLE OF A PROBLEM OF ALGEBRA.

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Abstract:

In this work we will present an approach from three perspectives of themes / topical issues and, unfortunately, very little known pupils, students and teachers: determination of some isomorphisms of algebraic structures. As applications, we will determine a series of real numbers fields isomorphic to the field ($\mathbf{R}$,+,·). Part of the results will be fully proven, and a part will be left to the reader interisd in these issues.

Introduction:-

The problem / theme of the determination of the different isomorphic algebra structures, respectivelyely the determination of the isomorphisms of such structures is quite a difficult one. Depending on their degree of difficulty, we can say that there are three categories of such issues. Thus, two algebraic structures A and B of the same type (groupoids, semigroups, monoids, groups, rings or fields) are given, in the process of teaching – learning the following types of problems may occur:

Problem 1: Prove that a function:

$$f : A \to B$$

is an isomorphism of such structures; or, in other words, to show that structures A and B are isomorphic by isomorphism f.

In this case may be required to determine one or more conditions for the function f to be an isomorphism; that is, for structures A and B to be isomorphic.

Problem 2: Show that structures A and B are isomorphic.

In this problem it is necessary to determine the isomorphism between structures A and B or it is required to prove its existence (not necessarily highlighting it).

Problem 3: Are structures A and B are isomorphic?

In this third case, the problem is one of decidability: if structures A and B are isomorphic, then it must be justified why they are isomorphous, and if they are not isomorphic, then we need to show why this is happening.

In this paper we will present from the triple perspective (inductive, deductive and analogous) a problem of determining of some fields of real numbers isomorphic to the field ($\mathbf{R}$,+.·). In this way, we will show, for the beginning, that any open interval of real numbers, bordered or unlimited, can be endowed with a (commutative) field structure isomorphic to the (commutative) field of real numbers ($\mathbf{R}$,+,·). Then we will present a series of ten applications of this, effectively determining such isomorphic structures.

As I said above, we will address this issue from an inductive, deductive and analogical perspective.

The paper is intended to be one of the Didactics of Mathematics (both school and university), more precisely an applied didactics. It addresses, alike, pupils, students and teachers, that is to say, to all those who wish to increase their knowledge baggage in this area, to form and develop the methodological competences to solve this type of problems, in order to better prepare classroom lessons, of participation at competitions and Olympiads or of professional development.

Of course, for easy reafromg of this paper, we assume that the reader is familiar with all the notions presented here.

Technical results:-

In this paragraph we will present, with complete proofs, a series of theoretical results particularly useful further.

The main result of this work is the following:

Theorem 2.1: *If $(G,\cdot)$ is a group (commutative) and H is a nonempty set, and*:

$$f : G \to H$$

is a bijective function, then the map:

$$„\circ" : H\times H \to H,$$

defined by: for every $(x,y)\in H\times H$,

$$x\circ y=f(f^{-1}(x)\cdot f^{-1}(y)) \tag{2.1}$$

is an internal composition law (group operation) on H and which determines on H a structure of group (commutative), isomorphic to the group $(G,\cdot)$.

Proof: Let be x, $y\in H$. Then, according to the hypothesis, $f^{-1}(x)$, $f^{-1}(y)$ and $f^{-1}(x)\cdot f^{-1}(y)\in G$. It follows that:

$$x\circ y=f(f^{-1}(x)\cdot f^{-1}(y))\in H$$

and, thus, the map „$\circ$" is an internal composition law on the set H. We consider, now, three elements x, y and z from H. Then, the following equalities hold:

(1) $(x\circ y)\circ z=f(f^{-1}(x)\cdot f^{-1}(y))\circ z$
$=f(f^{-1}(f(f^{-1}(x)\cdot f^{-1}(y))\cdot f^{-1}(z))$
$=f((f^{-1}(x)\cdot f^{-1}(y))\cdot f^{-1}(z))$
$=f(f^{-1}(x)\cdot (f^{-1}(y)\cdot f^{-1}(z)))$
$=f(f^{-1}(x)\cdot f^{-1}(f(f^{-1}(y)\cdot f^{-1}(z))))$
$=f(f^{-1}(x)\cdot f^{-1}(y\circ z))$
$=x\circ(y\circ z)$.

Retaining the extremities from the equalities (1), obtain that the law „$\circ$" is associative. If the law „$\cdot$" is commutative on G, then, for every x, $y\in H$:

(2) $x\circ y=f(f^{-1}(x)\cdot f^{-1}(y))$
$=f(f^{-1}(y)\cdot f^{-1}(x))$
$=y\circ x$,

from where it follows that (and) the law „$\circ$" is commutative. On the other hand, let be $e_G\in G$ – the identity element towards the law „$\cdot$" and $e_H\in H$, for which, whatever it is $x\in H$,

(3) $x\circ e_H=x$.

The equality (3) is equivalent to:

(4) $f(f^{-1}(x)\cdot f^{-1}(e_H))=x$,

equality that (by applying f^{-1}) it occurs exactly when:

(5) $f^{-1}(x)\cdot f^{-1}(e_H)=f^{-1}(x)$.

This last equality is seen in the group G, where we can "*simplify*" with any element. It follows that:

(6) $f^{-1}(e_H)=e_G$,

that is:

$$e_H=f(e_G). \tag{2.2}$$

Therefore, the element $e_H\in H$, defined by the equality (2.2) is the identity element towards the law „$\circ$". Finally, we show that every element $x\in H$ has a symmetric (an inverse) relative to the law „$\circ$". Therefore, let be x, $x_H^{-1}\in H$ thus încât:

(7) $x\circ x_H^{-1}=f(e_G)$.

From the equalities (2.1) and (8), it follows that:

(8) $f(f^{-1}(x)\cdot f^{-1}(x_H^{-1}))=f(e_G)$.

Because, according to the hypothesis, f is injective, from the equality (8), it follows that:

(9) $f^{-1}(x)\cdot f^{-1}(x_H^{-1})=e_G$.

So:

(10) $f^{-1}(x_H^{-1})=(f^{-1}(x))_G^{-1}\in G$,

that is:

$$x_H^{-1}=f(f^{-1}(x))_G^{-1}). \tag{2.3}$$

Therefore, every element $x\in H$, has in H, a symmetrical x_H^{-1} given by the equality (2.3), where $(f^{-1}(x))_G^{-1}$ is the inverse of the element $f^{-1}(x)$ in the group G. Further we will show that f is (even) the isomorphism between the two groups. Because, according to the hypothesis, f is bijective, we only have to show that f is the (homo)morphism of groups. Thus, let be g_1, $g_2\in G$. Because, according to the same hypotheses, f^{-1} is surjective, there are x, $y\in H$, such that:

$g_1=f^{-1}(x)$ and $g_2=f^{-1}(y)$.

Then,

$f(g_1)=x$, $f(g_2)=y$

and

(11)
$$\begin{aligned} f(g_1\cdot g_2)&=f(f^{-1}(g_1)\cdot f^{-1}(g_2))\\ &=x\circ y\\ &=f(g_1)\circ f(g_2). \end{aligned}$$

Now the theorem is completely proved. □

Now, obtain, immediately, the following result:

Corollary 2.2: *If $(G,\cdot)$ is a (commutative) group, and H and K are two equipotens sets with G, then there are two laws of internal composition*:

$$\text{„}\circ\text{”} : H\times H \to H, \tag{2.4}$$

respectively

$$\text{„}\bullet\text{”} : K\times K \to K, \tag{2.5}$$

which determines on H, respectively on K, a structure of (commutative) group, each isomorph to the group $(G,\cdot)$, such that the following diagram (of isomorphic groups) is commutative:

$$\begin{tikzcd} (G,\cdot) \arrow[r] \arrow[d] & (H,\circ) \arrow[dl] \\ (K,\bullet). & \end{tikzcd} \tag{2.6}$$

Proof: According to the hypothesis, there are the following bijective functions:

$f : G \to H$ and $g : G \to K$.

From Theorem 2.1 it follows that the maps (2.4) and (2.5) defined by: for every x, $y\in H$,

(1) $x\circ y=f(f^{-1}(x)\cdot f^{-1}(y))\in H$,

respectively, for every x, $y\in K$,

(2) $x\bullet y=g(g^{-1}(x)\cdot g^{-1}(y))\in K$,

are laws of internal composition on H, respectively on K, which determines each from these sets a structure of (commutative) group. Moreover, we consider the function:

(3) $h=g\circ f^{-1} : H \to K$.

Because, according to the hypothesis, f is bijective, she is invertible and its inverse:

$f^{-1} : H \to G$

it is also bijective. After that, because composed of two bijective functions is, in turn, also a bijective function, also a bijective function, it follows that the function h, as defined above, is bijective. On the other hand, because, according to the proof of Theorem 2.1, f is morphism of groups and its inverse f^{-1} it is also morphism of groups. Finally, because the composition of two morphisms of groups is, in turn, a morphism of groups, it follows that the function h, as defined above, is also a morphism of groups. Therefore, through the map h, the groups $(H,\circ)$ and $(K,\bullet)$ are isomorphic and the diagram in the statement is commutative. □

The following remarks are required here:

Remarks 2.3: ***1)*** *The fact that the function h, from the proof of Corollary 2.2, is a morphism of groups it can be shown directly*.

2) *From the proof of Corollary 2.2 it follows that the function f, which determines the compositional law on the set*

H, we can also consider it from H to G; and in this case we obtain the same composition law on H as in the case of Theorem 2.2.

Proof**: 1)** Indeed, let be x, $y\in H$. Then, from the equalities (1), (2) and (3) from the proof of Corollary 2.2, obtain that:

$$\begin{aligned} h(x\circ y) &= (g\circ f^{-1})(x\circ y) \\ &= g(f^{-1}(x\circ y)) \\ &= g(f^{-1}(f(f^{-1}(x)\cdot f^{-1}(y)))) \\ &= g((f^{-1}(x)\cdot f^{-1}(y))) \\ &= g(f^{-1}(x))\circ g(f^{-1}(y))) \\ &= h(x)\circ h(y). \end{aligned}$$

2) If:

$$f : H \to G$$

is a bijective function, then the map:

(1) $x\perp y=f^{-1}(f(x)\cdot f(y))$

is an internal composition law on H and which determines on H a structure of (commutative) group, isomorph to the (commutative) group $(G,\cdot)$. To show this, the proof of Theorem 2.1 restores, replacing f with f^{-1} and invers. Thus, let be x, $y\in H$. Then, according to the hypothesis, f(x), f(y) and $f(x)\cdot f(y)\in G$. It follows that:

$$x\perp y=f^{-1}(f(x)\cdot f(y))\in H$$

and, thus, the map „$\perp$" is an internal composition law on the set H. We consider, now, three elements x, y and z belonging to H. Then:

(2) $$\begin{aligned} (x\perp y)\perp z &= f^{-1}(f(x)\cdot f(y))\perp z \\ &= f^{-1}(f(f^{-1}(f(x)\cdot f(y))\cdot f(z)) \\ &= f^{-1}((f(x)\cdot f(y))\cdot f(z)) \\ &= f^{-1}(f(x)\cdot (f(y)\cdot f(z))) \\ &= f^{-1}(f(x)\cdot f(f^{-1}(f(y)\cdot f(z)))) \\ &= f^{-1}(f(x)\cdot f(y\perp z)) \\ &= x\perp(y\perp z). \end{aligned}$$

Retaining the extremities from the equalities (2), obtain that the law „$\perp$" is associative. If the law „$\cdot$" is commutative on G, then, for every x, $y\in H$:

(3) $$\begin{aligned} x\perp y &= f^{-1}(f(x)\cdot f(y)) \\ &= f^{-1}(f(y)\cdot f(x)) \\ &= y\perp x, \end{aligned}$$

whence it follows that the law „$\perp$" is (also) commutative. On the other hand, let be $e_G\in G$ – the identity element towards the law „$\cdot$" and $e_H\in H$, for which, whatever it is $x\in H$,

(4) $x\perp e_H=x$.

The equality (4) is equivalent to:

(5) $f^{-1}(f(x)\cdot f(e_H))=x$,

equality which (by applying f) holds exactly if:

(6) $f(x)\cdot f(e_H)=f(x)$.

This last equality is seen in group G, where we can "*simplify*" with any element. It follows that:

(7) $f(e_H)=e_G$,

that is:

$$e_H=f^{-1}(e_G). \tag{2.7}$$

Therefore, the element $e_H\in H$, defined by the equality (2.7) is the identity element towards the law „$\perp$". Finally, we will show that each element $x\in H$ has a symmetrical (inverse) relationship with the law „$\perp$". Therefore, let be x, $x_H^{-1}\in H$ such that:

(8) $x\perp x_H^{-1}=f^{-1}(e_G)$.

From the equalities (1) and (8), it follows that:

(9) $f^{-1}(f(x)\cdot f(x_H^{-1}))=f^{-1}(e_G)$.

Because, according to the hypothesis, f^{-1} is injective, from the equality (9), it follows that:

(10) $f(x)\cdot f(x_H^{-1})=e_G$.

So:

(11) $f(x_H^{-1})=(f(x))_G^{-1}\in G$,

that is:

$$x_H^{-1}=f^{-1}((f(x))_G^{-1}). \tag{2.8}$$

Therefore, each element $x\in H$, has in H, a symmetric (inverse) x_H^{-1} given by the equality (2.8), where $(f(x))_G^{-1}$ represents the inverse of the element f(x) in the group G. Next, we will show that f is (even) the isomorphism between the two groups. Because, according to the hypothesis, f^{-1} is bijective, we only have to show that f^{-1} is a morphism (from $(G,\cdot)$ to $(H,\perp)$). Thus, let be g_1, $g_2\in G$. Because, according to the same hypotheses, f is surjective, it follows that there are x, $y\in H$, such that:

$g_1=f(x)$ and $g_2=f(y)$.

Then,

$f^{-1}(g_1)=x$, $f^{-1}(g_2)=y$

and

(12) $f^{-1}(g_1\cdot g_2)=f^{-1}(f(x)\cdot f(y))$
$=x\perp y$
$=f^{-1}(g_1)\perp f^{-1}(g_2)$.

It's very easy to show that and in this case, f is a morphism from the group $(H,\perp)$ to the group $(G,\cdot)$. Indeed, if x, $y\in H$, then, according to equality (1) or by applying the equivalences (12), obtain that:

$f(x\perp y)=f(f^{-1}(f(x)\cdot f(y)))$
$=f(x)\cdot f(y)$.

Now and the second remark is completely proven. □

Now, we present the following result, which will be useful to us in future developments:

Lemma 2.4: *Let G and H be two nonempty sets, „+" and „·" - two laws of internal composition on the set G, and*:

$f: G\to H$

is a bijective function. If the law „·" is distributive to the left (right) to the law „+", then and the law:

„•" : $H\times H\to H$,

defined by: for every $(x,y)\in H\times H$,

$$x\bullet y=f(f^{-1}(x)\cdot f^{-1}(y)) \tag{2.9}$$

is distributive to the left (right) to the law:

„○" : $H\times H\to H$,

defined by: for every $(x,y)\in H\times H$,

$$x\circ y=f(f^{-1}(x)+f^{-1}(y)). \tag{2.10}$$

Proof**:** Let be x, y, $z\in H$. Then:

$$\begin{aligned}(x\circ y)\bullet z&=f(f^{-1}(x)+f^{-1}(y))\bullet z\\&=f(f^{-1}(f(f^{-1}(x)+f^{-1}(y)))\cdot f^{-1}(z))\\&=f((f^{-1}(x)+f^{-1}(y))\cdot f^{-1}(z))\\&=f(f^{-1}(x)\cdot f^{-1}(z)+f^{-1}(y)\cdot f^{-1}(z))\\&=f(f^{-1}(x\bullet z)+f^{-1}(y\bullet z))\\&=(x\bullet z)\circ(y\bullet z).\end{aligned}$$

Therefore, the law „•" is distributive to the right to the law „○". The left-hand distributivity is also shown analogously. □

From Lemma 2.4 and Remarks 2.3, point 2), we immediately deduce:

Remark2.5: *The statement in Lema 2.4 remains valid and if we consider the function f from H to G*. □

Now, from Theorem 2.1 and Lema 2.4, obtain:

Corollary 2.6: *If (A,+,·) is a (commutative) ring / field and B is a set equipotent with the set A, then there are two laws of internal composition on B, which we note with „○" and „•", such that (R,○,•) is a (commutative) ring / field*. □

At the end of this paragraph we have:

Corollary 2.7: *If (A,+,·) is a (commutative) ring / field, and H and K are two sets equipotent with A, then, on each from the sets H and K, there are two laws of internal composition which determines on H, respectively on K, a structure of (commutative) ring / field, each isomorph to the ring / the field A, such that the following diagram (of isomorphic rings / fields) is commutative*:

$A \longrightarrow H$

$$\downarrow \quad \swarrow \tag{2.11}$$

K. □

Applications: Isomorphic fields to the field (R,+,·)

In this paragraph we will present ten applications of the ones shown in the previous paragraph, in the sense that we will construct / deduce three types of fields of real numbers, isomorphic to the field ($\mathbf{R}$,+,·). The first application will be fully proved, and the others, can be demonstrated by analogy, we propose them for proof reader's careful and interested in these issues.

In this way, let $\mathbf{R}$ be – the set of real numbers and a, b, c, d, m, n, p, q$\in\mathbf{R}$. Also, we consider the following sets / intervals of real numbers:

$$(-\infty,a)=\{x\in\mathbf{R} \mid x<a\} \overset{\text{not}}{=} \mathbf{R}_{-\infty,a},$$

$$(a,b)=\{x\in\mathbf{R} \mid a<x<b\} \overset{\text{not}}{=} \mathbf{R}_{a,b}$$

and

$$(a,+\infty)=\{x\in\mathbf{R} \mid x<a\} \overset{\text{not}}{=} \mathbf{R}_{a,+\infty}.$$

In this paragraph we will show that on each of the three intervals we can define two laws of internal composition which will determine the structure of the (commutative) field. The fields thus obtained will be isomorphic to each other and each will be isomorphic to the real numbers field, ($\mathbf{R}$,+,·).

3.1 Fields of the form ($\mathbf{R}_{-\infty,a}$,○,●) isomorphic to the field (R,+,·)

***Application 3.1.1*: *For every a$\in$**R**, there are two laws of internal composition on the set*:

$$\mathbf{R}_{-\infty,a}=(-\infty,a),$$

*which to determine on this interval a structure of field (commutative) isomorphic to the field (**R**,+,·).*

***Proof*:** We consider the field ($\mathbf{R}$,+,·), the interval of real numbers $\mathbf{R}_{-\infty,a}$ and the function:

$$f_1 : \mathbf{R} \to \mathbf{R}_{-\infty,a},$$

defined by: for every x$\in\mathbf{R}$,

$$f_1(x)=a-e^{-x}. \tag{3.1.1}$$

We show that the function f_1 is injective. In this way, we consider x, y$\in\mathbf{R}$, such that:

$$f_1(x)=f_1(y).$$

It follows that:

$$a-e^{-x}=a-e^{-y},$$

where from, because the function e^{-x} is injective, it follows that:

$$x=y$$

and, thus, the function f_1 is injective. To prove the surjectivity of f_1, we consider any element y$\in\mathbf{R}_{-\infty,a}$ and we solve the equation:

$$f_1(x)=y.$$

This equation becomes:

$$a-e^{-x}=y,$$

where from it follows that:

$$e^{-x}=a-y,$$

that is:

$$-x=\ln(a-y)\in\mathbf{R},$$

because, according to the choice of y, a-y>0. Therefore, f_1 is bijective, and:

$$f_1^{-1} : \mathbf{R}_{-\infty,a} \to \mathbf{R}$$

is defined by: for every x, y$\in\mathbf{R}_{-\infty,a}$,

$$f_1^{-1}(x)=-\ln(a-x)$$

$$=\ln\frac{1}{a-x}. \tag{3.1.2}$$

According to Theorem 2.1, Lemma 2.4 and Corollary 2.6, the maps:

$$\text{„}\circ\text{”} : \mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,a},$$

defined by: for every x, y$\in\mathbf{R}_{-\infty,a}$,

$$x\circ y=f_1(f_1^{-1}(x)+f_1^{-1}(y)), \tag{3.1.3}$$

and

$$„\bullet” : \mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a}\to\mathbf{R}_{-\infty,a},$$

defined by: for every x, y$\in\mathbf{R}_{-\infty,a}$,

$$x\bullet y=f_1(f_1^{-1}(x)\cdot f_1^{-1}(y)), \tag{3.1.3′}$$

there are two laws of composition on $\mathbf{R}_{-\infty,a}$, which determine on this set a structure of (commutative) field isomorphic to the field $(\mathbf{R},+,\cdot)$. In our case, the equality (3.1.3) becomes:

$$\begin{aligned} x\circ y&=f_1(-\ln[(a-x)(a-y)]) \\ &=a-e^{\ln[(a-x)(a-y)]} \\ &=a-(a-x)(a-y) \\ &=a-a^2+a(x+y)-xy, \end{aligned} \tag{3.1.4}$$

and the equality (3.1.3′) becomes:

$$\begin{aligned} x\bullet y&=f_1([\ln(a-x)\cdot\ln(a-y)]) \\ &=a-e^{-[\ln(a-x)\cdot\ln(a-y)]} \\ &=a-(a-x)^{-\ln(a-y)} \\ &=a-(a-y)^{-\ln(a-x)}. \end{aligned} \tag{3.1.4′}$$

The neutral (identity) element of the group $(\mathbf{R}_{-\infty,a},\circ)$ is, according to equality (2.2),

$$\begin{aligned} e_{R_{-\infty,a}}&=f(0) \\ &=a-1\in\mathbf{R}_{-\infty,a}, \end{aligned} \tag{3.1.5}$$

and the symmetric on an element $x\in\mathbf{R}_{-\infty,a}$, relative to the law „$\circ$” is, according to equality (2.3) and the equalities (3.1.1) and (3.1.2) above, the element:

$$\begin{aligned} x_{R_{-\infty,a}}^{-1}&=f_1(-f_1^{-1}(x)) \\ &=f_1(\ln(a-x)) \\ &=a-e^{-\ln(a-x)} \\ &=a-e^{\ln\frac{1}{a-x}} \\ &=a-\frac{1}{a-x}\in\mathbf{R}_{-\infty,a}. \end{aligned} \tag{3.1.6}$$

The neutral (identity) element of the group $(\mathbf{R}_{-\infty,a}\backslash\{a-1\},\bullet)$ is, according to equality (2.2),

$$\begin{aligned} e_{R_{-\infty,a}\backslash\{a-1\}}&=f_1(1) \\ &=a-e^{-1}\in\mathbf{R}_{-\infty,a\backslash\{a-1\}}, \end{aligned} \tag{3.1.5′}$$

and the symmetric on an element $x\in\mathbf{R}_{-\infty,a}\backslash\{a-1\}$, relative to the law „$\bullet$” is, according to equality (2.3) and the equalities (3.1.1) and (3.1.2) above, the element:

$$\begin{aligned} x_{R_{-\infty,a}\backslash\{a-1\}}^{-1}&=f_1((f_1^{-1}(x))^{-1}) \\ &=a-e^{\frac{1}{\ln(a-x)}}\in\mathbf{R}_{-\infty,a}\backslash\{a-1\}. \end{aligned} \tag{3.1.6′}$$

Now, we verify that the law „$\bullet$” is distributive to the law „$\circ$”. Indeed, if x, y, z$\in\mathbf{R}_{-\infty,a}$, then, according of the above, we have:

$$\begin{aligned} (x\circ y)\bullet z&=[a-(a-x)\cdot(a-y)]\bullet z \\ &=\alpha\bullet z \\ &=a-(a-\alpha)^{-\ln(a-z)} \\ &=a-[(a-x)\cdot(a-y)]^{-\ln(a-z)} \\ &=a-(a-x)^{-\ln(a-z)}\cdot(a-y)^{-\ln(a-z)} \\ &=a-\{a-[a-(a-x)^{-\ln(a-z)}]\}\cdot\{a-[a-(a-y)^{-\ln(a-z)}]\} \\ &=[a-(a-x)^{-\ln(a-z)}]\circ[a-(a-y)^{-\ln(a-z)}] \\ &=(x\bullet z)\circ(y\bullet z); \end{aligned}$$

which shows distributivity to the right of the law „$\bullet$” to the law „$\circ$”; where:

$$\alpha=a-(a-x)\cdot(a-y).$$

The left-hand distributivity is also shown analogously. At the end of this proof, let us verify that, indeed, as we have shown in the proof of Theorem 2.1, the function f_1 is also an isomorphism of fields. Thus, we consider x, y$\in$**R**. Then, from the equality (3.1.1), obtain that:

$$f_1(x+y)=a-e^{-(x+y)}, \tag{3.1.7}$$

and, from the equalities (3.1.4) obtain that:

$$\begin{aligned} f_1(x)\circ f_1(y)&=a-(a-f_1(x))(a-f_1((y)) \\ &=a-(a-a+e^{-x})(a-a+e^{-y}) \\ &=a-e^{-(x+y)}. \end{aligned} \tag{3.1.8}$$

From the equalities (3.1.7) and (3.1.8) it follows that:

$$f_1(x+y)=f_1(x)\circ f_1(y). \tag{3.1.9}$$

Analogous we also obtain the equalities:

$$f_1(x\cdot y)=a-e^{-x\cdot y}, \tag{3.1.7′}$$

and, from the equalities (3.1.4) obtain that:

$$\begin{aligned} f_1(x)\bullet f_1(y)&=a-(a-f_1(x))^{-\ln(a-f_1(y))} \\ &=a-(a-a+e^{-x})^{-\ln(a-a+e^{-y})} \\ &=a-(e^{-x})^{-\ln(e^{-y})} \\ &=a-(e^{-x})^{y} \\ &=a-e^{-xy}. \end{aligned} \tag{3.1.8′}$$

From the equalities (3.1.7′) and (3.1.8′) it follows that:

$$f_1(x+y)=f_1(x)\bullet f_1(y). \tag{3.1.9′}$$

Therefore, according to the equalities (3.1.9) and (3.1.9′), the function f_1 is (also) a morphism of fields, from the field (**R**,+,·) to the field ($\mathbf{R}_{-\infty,a}$,○,●).

Let's notice that the bijectivity of f_1 we can also show it using mathematical analysis. Thus, for every x$\in$**R**,

$$f_1'(x)=e^{-x}>0, \tag{3.1.10}$$

which shows that f_1 is strictly increasing on **R**. Being (also) continuously on **R**, it follows that f_1 is injective. On the other hand,

$$\lim_{x\to-\infty} f_1(x)=-\infty, \tag{3.1.11}$$

$$\lim_{x\to\infty} f_1(x)=a. \tag{3.1.12}$$

Because f_1 has the Darboux property on **R** (being continuous), it follows that f_1 is surjective. So f_1 is bijective. □

The following remark are required here:

Remark 3.1.2: *As stated above, the function f_1 from Application 3.1.1 is increasing. We notice that we can also consider a function descending, for example*:

$$F_1 : \mathbf{R} \to (-\infty,a),$$

defined by: for every x$\in$***R***,

$$F_1(x)=a-e^{x}; \tag{3.1.1′}$$

the conclusion of Application 3.1.1 remaining the same.

Proof**:** Indeed, repeating the proof of Application 3.1.1 for the function F_1 we get the results below. We show that the function F_1 is injective. In this way we consider x, y$\in$**R**, such that:

$$F_1(x)=F_1(y).$$

It follows that:

$$a-e^{x}=a-e^{y},$$

where from, because the function e^x is injective, it follows that:

$$x=y$$

and, thus, the function F_1 is injective. To prove the surjectivity of F_1, we consider any element y$\in\mathbf{R}_{-\infty,a}$ and we solve the equation:

$$F_1(x)=y.$$

This equation becomes:

$$a-e^{x}=y,$$

where from it follows that:

$$e^{x}=a-y,$$

that is:

$$x=\ln(a-y)\in\mathbf{R},$$

because, according to the choice of y, a-y>0. Therefore, g_1 is bijective, and:

$F_1^{-1}: \mathbf{R}_{-\infty,a} \to \mathbf{R}$

is defined by: for every $x\in\mathbf{R}_{-\infty,a}$,

$$F_1^{-1}(x)=\ln(a-x) \qquad (3.1.2')$$

According to Theorem 2.1 and Lemma 2.4, the maps:

„∘" : $\mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,a}$,

defined by: for every $x, y\in\mathbf{R}_{-\infty,a}$,

$$x\circ y=F_1(F_1^{-1}(x)+F_1^{-1}(y)), \qquad (3.1.3')$$

and

„∗" : $\mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,a}$,

defined by: for every $x, y\in\mathbf{R}_{-\infty,a}$,

$$x*y=F_1(F_1^{-1}(x)\cdot F_1^{-1}(y)), \qquad (3.1.3'')$$

there are two laws of composition on $\mathbf{R}_{-\infty,a}$, which determine on this set a structure of (commutative) field isomorphic to the field $(\mathbf{R},+,\cdot)$. The equalities (3.1.3′) and (3.1.3″) implies:

$$\begin{aligned} x\circ y&=F_1(\ln[(a-x)(a-y)])\\ &=a-e^{\ln[(a-x)(a-y)]}\\ &=a-(a-x)(a-y)\\ &=a-a^2+a(x+y)-xy, \end{aligned} \qquad (3.1.4)$$

respectively

$$\begin{aligned} x*y&=F_1(\ln(a-x)\cdot\ln(a-y))\\ &=a-e^{\ln(a-x)\cdot\ln(a-y)}\\ &=a-(a-x)^{\ln(a-y)}\\ &=a-(a-y)^{\ln(a-x)}. \end{aligned} \qquad (3.1.4'')$$

Because the compositional laws „○" and „∘" coincides, we are only dealing with the second law, „∗". In this case, obtain:

$$\begin{aligned} e'_{R_{-\infty,a}} &=g_1(1)\\ &=a-e\in\mathbf{R}_{-\infty,a}, \end{aligned} \qquad (3.1.5'')$$

and the symmetric on an element $x\in\mathbf{R}_{-\infty,a}$, relative to the law „∗" is, according to equality (2.3) and the equalities (3.1.1′) and (3.1.2′) above, the element:

$$\begin{aligned} (x^{-1}_{R_{-\infty,a}\setminus\{a-1\}})'&=g_1((g_1^{-1}(x))^{-1})\\ &=g_1((\ln(a-x))^{-1})\\ &=a-e^{\frac{1}{\ln(a-x)}}\in\mathbf{R}_{-\infty,a}\setminus\{a-1\}. \end{aligned}$$

(3.1.6′)

Now, we verify that the law „∗" is distributive to the law „∘". Indeed, if $x, y, z\in\mathbf{R}_{-\infty,a}$, then, according of the above, we have:

$$\begin{aligned} (x\circ y)*z&=[a-(a-x)\cdot(a-y)]*z\\ &=\alpha*z\\ &=a-(a-\alpha)^{\ln(a-z)}\\ &=a-[(a-x)\cdot(a-y)]^{\ln(a-z)}\\ &=a-(a-x)^{\ln(a-z)}\cdot(a-y)^{\ln(a-z)}\\ &=a-\{a-[a-(a-x)^{\ln(a-z)}]\}\cdot\{a-[a-(a-y)^{\ln(a-z)}]\}\\ &=[a-(a-x)^{\ln(a-z)}]\circ[a-(a-y)^{\ln(a-z)}]\\ &=(x*z)\circ(y*z); \end{aligned}$$

which shows distributivity to the right of the law „∗" to the law „∘". And here

$$\alpha=a-(a-x)\cdot(a-y).$$

The left-hand distributivity is also shown analogously. At the end of this proof, let us verify that, indeed, as we have shown in the proof of Theorem 2.1, the function F_1 is also an isomorphism of fields. Thus, we consider $x, y\in\mathbf{R}$. Then, from the equality (3.1.1′), obtain that:

$$F_1(x+y)=a-e^{x+y}, \qquad (3.1.7'')$$

and, from the equalities (3.1.4) obtain that:

$$F_1(x)\circ F_1(y)=a-(a-F_1(x))(a-F_1((y))$$

$=a-(a-a+e^{x})(a-a+e^{y})$
$=a-e^{x+y}$. (3.1.8″)

From the equalities (3.1.7″) and (3.1.8″) it follows that:

$F_1(x+y)=F_1(x)\circ F_1(y)$. (3.1.9″)

Analogous we also obtain the equalities:

$F_1(x\cdot y)=a-e^{x\cdot y}$, (3.1.7‴)

and, from the equalities (3.1.4) obtain that:

$F_1(x)*F_1(y)=a-(a-g_1(x))^{\ln(a-g_1(y))}$
$=a-(a-a+e^{x})^{\ln(a-a+e^{y})}$
$=a-(e^{x})^{\ln(e^{y})}$
$=a-(e^{x})^{y}$
$=a-e^{x\cdot y}$. (3.1.8‴)

From the equalities (3.1.7‴) and (3.1.8‴) it follows that:

$F_1(x+y)=F_1(x)*F_1(y)$. (3.1.9‴)

Therefore, according to the equalities (3.1.9″) and (3.1.9‴), the function F_1 is (also) a morphism of fields, from the field $(\mathbf{R},+,\cdot)$ to the field $(\mathbf{R}_{-\infty,a},\circ,*)$.

Let's notice that the bijectivity of F_1 we can also show it using mathematical analysis. Thus, for every $x\in\mathbf{R}$,

$F_1'(x)=-e^{x}<0$, (3.1.10′)

which shows that F_1 is strictly decreasing on **R**. Being (also) continuously on **R**, it follows that F_1 is injective. On the other hand,

$\lim\limits_{x\to-\infty} F_1(x)=a$, (3.1.11′)

and

$\lim\limits_{x\to\infty} F_1(x)=-\infty$. (3.1.12′)

Because F_1 has the Darboux property on **R** (being continuous), it follows that the function F_1 is (also) surjective. So F_1 is bijective. □

Of course Application 3.1.1 is a deductive approach to our problem. Further to approach the problem of the isomorphism of these fields from an inductive perspective. In this way, let's solve it following problem:

Problem 3.1.3: *We consider $a\in\mathbf{R}$ and the following two laws of composition*:

„○" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

defined by: for every $(x,y)\in\mathbf{R}\times\mathbf{R}$,

$x\circ y=a-a^2+a(x+y)-xy$ (3.1.4)

and

„•" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

defined by: for every $(x,y)\in\mathbf{R}\times\mathbf{R}$,

$x\bullet y=a-(a-x)^{-\ln(a-y)}$. (3.1.4′)

Let's show that these operations determine on the set:

$\mathbf{R}_{-\infty,a}=(-\infty,a)$

a structure of (commutative) field and that the field $(\mathbf{R}_{-\infty,a},\circ,\bullet)$ is isomorphic to the field $(\mathbf{R},+,\cdot)$.

***Solution*:** For every x, $y\in\mathbf{R}_{-\infty,a}$, the equality (3.1.4) becomes:

$x\circ y=a-(a-x)(a-y)$. (3.1.4)

Because $(a-x)(a-y)>0$, it follows that $x\circ y<a$. Thus we have proven that $\mathbf{R}_{-\infty,a}$ is stable part (closed) of it **R** on the law „○". Show, now, that the law „○" is associative. Let be x, y, $z\in\mathbf{R}_{-\infty,a}$. Then:

(1) $(x\circ y)\circ z=[a-(a-x)(a-y)]\circ z$
$=\alpha\circ z$
$=a-(a-\alpha)(a-z)$
$=a-(a-x)(a-y)(a-z)$;

here:

$\alpha=a-(a-x)(a-y)$.

On the other hand,

(2) $x\circ(y\circ z)=x\circ[a-(a-y)(a-z)]$
$=x\circ\beta$

$=a-(a-x)(a-\beta)$
$=a-(a-x)(a-y)(a-z);$

here:

$\beta=a-(a-y)(a-z).$

From (1) and (2) it follows that „○” is associative. Because operations of addition and multiplication of real numbers are commutative, from the equality (3.1.4) it follows that this one is (also) commutative. We will determine, now, the neutral element of the law „○”. Thus, we determine the element $e_{R_{-\infty,a}}\in\mathbf{R}_{-\infty,a}$, such that, for every $x\in\mathbf{R}_{-\infty,a}$, to have the equality:

(3) $x\circ e_{R_{-\infty,a}}=x.$

This equality becomes:

(4) $a-(a-x)(a-e')=x,$

that is:

$(a-x)(a-e')=a-x.$

Because $x<a$, it follows that $a-x\neq0$ and, from last equality obtain that:

$$e_{R_{-\infty,a}}=a-1\in\mathbf{R}_{-\infty,a}. \tag{3.1.5}$$

To determine the inverse of an element $x\in\mathbf{R}_{-\infty,a}$, relative to the law „○”, denote with $x^{-1}_{R_{-\infty,a}}$ this element and we consider the equality:

(5) $x\circ x^{-1}_{R_{-\infty,a}}=a-1.$

This equality becomes:

(6) $a-(a-x)(a-x^{-1}_{R_{-\infty,a}})=a-1,$

what it implies:

$(a-x)(a-x^{-1}_{R_{-\infty,a}})=1,$

that is:

$$a-x^{-1}_{R_{-\infty,a}}=\frac{1}{a-x}.$$

From last equality obtain that:

(7) $x^{-1}_{R_{-\infty,a}}=a-\dfrac{1}{a-x}\in\mathbf{R}_{-\infty,a}.$

Therefore, we have shown that $(\mathbf{R}_{-\infty,a},\circ)$ is an abelian group. On the other hand, for every $x, y\in\mathbf{R}_{-\infty,a}\setminus\{a-1\}$ from the equality (3.2.4′) it follows that $x\bullet y<a$ and $a-x$, $a-y\in\mathbf{R}_{0,+\infty}\setminus\{1\}$. Thus we have proven that $\mathbf{R}_{-\infty,a}\setminus\{a-1\}$ is stable part (closed) of it $\mathbf{R}$ on the law „•”. Show, now, that the law „•” is associative. Let be $x, y, z\in\mathbf{R}_{-\infty,a}\setminus\{a-1\}$. Then:

(8) $(x\bullet y)\bullet z=[a-(a-x)^{-\ln(a-y)}]\circ z$
$=\gamma\bullet z$
$=a-(a-\gamma)^{-\ln(a-z)}$
$=a-(a-x)^{\ln(a-y)\cdot\ln(a-z)};$

here:

$\gamma=a-(a-x)^{-\ln(a-y)}.$

On the other hand,

(9) $x\bullet(y\bullet z)=x\bullet[a-(a-y)^{-\ln(a-z)}]$
$=x\bullet\delta$
$=a-(a-x)^{-\ln(a-\delta)}$
$=a-(a-x)^{\ln(a-y)\cdot\ln(a-z)};$

here:

$\delta=a-(a-y)^{-\ln(a-z)}.$

From (8) and (9) it follows that „•” is associative. Because, for every $s, t\in\mathbf{R}_{0,+\infty}$, the equality holds:

$s^{\ln t}=t^{\ln s},$

from the equality (3.1.4′), it follows that this is (also) commutative. We will determine, now, the neutral element of the law „•”. Thus, we determine the element $e_{R_{-\infty,a}\setminus\{a-1\}}\in\mathbf{R}_{-\infty,a}\setminus\{a-1\}$, such that, for every $x\in\mathbf{R}_{-\infty,a}\setminus\{a-1\}$, to have the equality:

(10) $x \circ e_{R_{-\infty,a}\setminus\{a-1\}} = x$.

This equality becomes:

(11) $a-(a-x)^{-\ln(a-e_{R_{-\infty,a}\setminus\{a-1\}})} = x$,

that is:

$$(a-x)^{-\ln(a-e_{R_{-\infty,a}\setminus\{a-1\}})} = a-x.$$

From last equality obtain that:

$$\ln(a-e_{R_{-\infty,a}\setminus\{a-1\}}) = -1,$$

that is

$$e_{R_{-\infty,a}\setminus\{a-1\}} = a-e^{-1} \in \mathbf{R}_{-\infty,a}\setminus\{a-1\}. \qquad (3.1.5')$$

To determine the inverse of an element $x \in \mathbf{R}_{-\infty,a}\setminus\{a-1\}$, relative to the law „•", denote with $x^{-1}_{R_{-\infty,a}\setminus\{a-1\}}$ this element and we consider the equality:

(12) $x \circ x^{-1}_{R_{-\infty,a}\setminus\{a-1\}} = a-e^{-1}$.

This equality becomes:

(13) $a-(a-x)^{-\ln(a-(x^{-1}_{R_{-\infty,a}})')} = a-e^{-1}$,

what it implies:

$$\ln(a-x)\cdot\ln(a-x^{-1}_{R_{-\infty,a}\setminus\{a-1\}}) = 1,$$

that is:

$$a-x^{-1}_{R_{-\infty,a}\setminus\{a-1\}} = e^{\frac{1}{\ln(a-x)}}.$$

From last equality obtain that:

(14) $x^{-1}_{R_{-\infty,a}\setminus\{a-1\}} = a-e^{\frac{1}{\ln(a-x)}} \in \mathbf{R}_{-\infty,a}\setminus\{a-1\}$.

Therefore, we have shown that $(\mathbf{R}_{-\infty,a}\setminus\{a-1\},\bullet)$ is an abelian group. The distributivity of law „•" to the law „○" is made as well as the proof of Application 3.2.1. To determine the isomorphism between the two fields we can consider the following function:

$$f_1 : \mathbf{R} \to \mathbf{R}_{-\infty,a},$$

defined by: for every $x \in \mathbf{R}$,

$$f_1(x) = a-e^{-x}. \qquad (3.1.1)$$

Now, we proceed as in the proof of Application 3.1.1, and we show that function f_1 is an isomorphism between the two fields. □

We notice that whatever the way to approach the isomorphism problem of the fields $(\mathbf{R}_{-\infty,a},\circ,\bullet)$ and $(\mathbf{R},+,\cdot)$, the results obtained are the same.

3.2 Fields of the form $(\mathbf{R}_{b,c},\circ,\bullet)$ isopmorphic to the field $(\mathbf{R},+,\cdot)$

Application 3.2.1**:** *For every b, c* $\in \mathbf{R}$*, cu b<c, there are two laws of internal composition on the set*:

$\mathbf{R}_{b,c}=(b,c)$*,*

which to determine on this interval a structure of (commutative) field isomorphic to the field $(\mathbf{R},+,\cdot)$.

Hint**:** We consider the field $(\mathbf{R},+,\cdot)$, the interval of real numbers:

$$\mathbf{R}_{b,c}=(b,c)$$

and the function:

$$f_2 : \mathbf{R} \to \mathbf{R}_{b,c},$$

defined by: for every $x \in \mathbf{R}$,

$$f_2(x) = \frac{be^x + c}{1+e^x}. \qquad (3.2.1)$$

It shows that the function f_2 is bijective and:

$f_2^{-1}: \mathbf{R}_{b,c} \to \mathbf{R}$

is defined by: for every x, y∈$\mathbf{R}_{b,c}$,

$$f_2^{-1}(x)=\ln\frac{c-x}{x-b}. \qquad (3.2.2)$$

According to those presented in Paragraph 2, the required maps are:

„○" : $\mathbf{R}_{b,c}\times\mathbf{R}_{b,c} \to \mathbf{R}_{b,c}$,

defined by: for every x, y∈$\mathbf{R}_{b,c}$,

$$x\circ y=f_2(f_2^{-1}(x)+f_2^{-1}(y))$$
$$=\frac{xy(b+c)-2bc(x+y)+bc(b+c)}{2xy-(b+c)(x+y)+b^2+c^2} \qquad (3.2.3)$$

and

„•" : $\mathbf{R}_{b,c}\times\mathbf{R}_{b,c} \to \mathbf{R}_{b,c}$,

defined by: for every x, y∈$\mathbf{R}_{b,c}$,

$$x\bullet y=f_2(f_2^{-1}(x)\cdot f_2^{-1}(y))$$
$$=\frac{b\left(\frac{c-x}{x-b}\right)^{\ln\left(\frac{c-y}{y-b}\right)}+c}{1+\left(\frac{c-x}{x-b}\right)^{\ln\left(\frac{c-y}{y-b}\right)}}. \qquad (3.2.4)$$

Reasoning as in the Application 3.1.1 it shows that ($\mathbf{R}_{b,c}$,○,•) is a commutative field isomorphic to ($\mathbf{R}$,+,·). □

The following remark are required here:

Remark 3.2.2: *The function f_2 from Application 3.2.1 is decreasing. Observe that we can also consider it an increasing function, for example*:

$F_2 : \mathbf{R} \to \mathbf{R}_{b,c}$,

*defined by: for every x∈**R**,*

$$F_2(x)=\frac{b+ce^x}{1+e^x}; \qquad (3.2.1')$$

the conclusion of Application 3.2.1 remaining the same. □

The inductive approach of this isomorphism of fields involves solving the problem:

Problem 3.2.3: *We consider b, c∈**R**, cu b<c and the following two laws of composition*:

„○" : $\mathbf{R}\times\mathbf{R} \to \mathbf{R}$,

*defined by: for every (x,y)∈**R**×**R**,*

$$x\circ y=\frac{xy(b+c)-2bc(x+y)+bc(b+c)}{2xy-(b+c)(x+y)+b^2+c^2} \qquad (3.2.3)$$

and

„•" : $\mathbf{R}\times\mathbf{R} \to \mathbf{R}$,

*defined by: for every (x,y)∈**R**×**R**,*

$$x\bullet y=\frac{b\left(\frac{c-x}{x-b}\right)^{\ln\left(\frac{c-y}{y-b}\right)}+c}{1+\left(\frac{c-x}{x-b}\right)^{\ln\left(\frac{c-y}{y-b}\right)}}. \qquad (3.2.4)$$

Prove that these operations determines on the set:

$\mathbf{R}_{b,c}=(b,c)$

*a structure of commutative field and that ($\mathbf{R}_{b,c}$,○,•) is a field isomorphic to the field (**R**,+,·).* □

3.3 Fields of the form ($\mathbf{R}_{d,+\infty}$,○,•) isopmorphic to the field (R,+,·)

Application 3.3.1: *For every d∈**R**, there are two laws of internal composition on the set*:

$\boldsymbol{R}_{d,+\infty}=(d,+\infty)$,

which to determine on this interval a structure of (commutative) field isomorphic to the field ($\boldsymbol{R}$,+,·).

***Hint*:** We consider the field ($\mathbf{R}$,+,·), the interval of real numbers:

$\mathbf{R}_{d,+\infty}=(d,+\infty)$

and the function:

$f_3 : \mathbf{R} \to \mathbf{R}_{d,+\infty}$,

defined by: for every $x\in\mathbf{R}$,

$$f_3(x)=d+e^x. \qquad (3.3.1)$$

It shows that the function f_3 is bijective and:

$f_3^{-1} : \mathbf{R}_{d,+\infty} \to \mathbf{R}$

is defined by: for every x, $y\in\mathbf{R}_{d,+\infty}$,

$$f_3^{-1}(x)=\ln(x-d). \qquad (3.3.2)$$

According to those presented in Paragraph 2, the required maps are:

„○" : $\mathbf{R}_{d,+\infty}\times\mathbf{R}_{d,+\infty} \to \mathbf{R}_{d,+\infty}$,

defined by: for every x, $y\in\mathbf{R}_{d,+\infty}$,

$$x\circ y=f_3(f_3^{-1}(x)+f_3^{-1}(y))$$
$$=d+(x-d)(y-d) \qquad (3.3.3)$$

and

„•" : $\mathbf{R}_{d,+\infty}\times\mathbf{R}_{d,+\infty} \to \mathbf{R}_{d,+\infty}$,

defined by: for every x, $y\in\mathbf{R}_{d,+\infty}$,

$$x\bullet y=f_3(f_3^{-1}(x)\cdot f_3^{-1}(y))$$
$$=d+(x-d)^{\ln(y-d)}. \qquad (3.3.4)$$

Reasoning as in the Application 3.1.1 it shows that ($\mathbf{R}_{d,+\infty}$,○,•) is a commutative field isomorphic to ($\mathbf{R}$,+,·). □

The following remark are required here:

Remark 3.3.2: *The function f_3 from Application 3.3.1 is increasing. Observe that we can also consider it an decreasing function, for example*:

$F_3 : \boldsymbol{R} \to \boldsymbol{R}_{d,+\infty}$,

defined by: for every $x\in\boldsymbol{R}$,

$$F_3(x)=d+e^{-x}; \qquad (3.3.1')$$

the conclusion of Application 3.3.1 remaining the same. □

The inductive approach of this isomorphism of fields involves solving the problem:

Problem 3.3.3: *We consider $d\in\boldsymbol{R}$ and the following two laws of composition*:

„○" : $\boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$,

defined by: for every $(x,y)\in\boldsymbol{R}\times\boldsymbol{R}$,

$$x\circ y=d+(x-d)(y-d) \qquad (3.3.3)$$

and

„•" : $\boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$,

defined by: for every $(x,y)\in\boldsymbol{R}\times\boldsymbol{R}$,

$$x\bullet y=d+(x-d)^{\ln(y-d)}. \qquad (3.3.4)$$

Prove that these operations determines on the set:

$\boldsymbol{R}_{d,+\infty}=(d,+\infty)$

a structure of commutative field and that ($\boldsymbol{R}_{d,+\infty}$,○,•) is a field isomorphic to the field ($\boldsymbol{R}$,+,·). □

3.4 Fields of the form ($\mathbf{R}_{-\infty,a}$,○) isomorphic between them (but also to the field ($\mathbf{R}$,+,·))

Application 3.4.1: *For every a, $m\in\boldsymbol{R}$, there are two laws of internal composition on the set*:

$\boldsymbol{R}_{-\infty,a}=(-\infty,a)$

and there are two laws of internal composition on the set:

$\boldsymbol{R}_{-\infty,m}=(-\infty,m)$,

which to determine on these intervals a structure of (commutative) field and the two fields to be isomorphic between them, but also to the field ($\boldsymbol{R}$,+,·).

***Hint*:** Observe that this application is a particular case of Corollary 2.2, for:

$\mathbf{R}_{-\infty,a}=(-\infty,a)$ and $\mathbf{R}_{-\infty,m}=(-\infty,m)$.

Therefore, according to Application 3.1.1, there are the bijective functions:

$u : \mathbf{R} \to \mathbf{R}_{-\infty,a}$ and $v : \mathbf{R} \to \mathbf{R}_{-\infty,m}$,

defined by: for every $x \in \mathbf{R}$,

$$u(x)=a-e^{-x}, \tag{3.1.1}$$

respectively,

$$v(x)=m-e^{-x}, \tag{3.1.1″}$$

which determines the following maps:

$„\circ” : \mathbf{R}_{-\infty,a} \times \mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,a}$ and $„\bullet” : \mathbf{R}_{-\infty,a} \times \mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,a}$,

defined by: for every $x, y \in \mathbf{R}_{-\infty,a}$,

$$x \circ y=u(u^{-1}(x)+u^{-1}(y)) \in \mathbf{R}_{-\infty,a}, \tag{3.1.3‴}$$

$$x \bullet y=u(u^{-1}(x) \cdot u^{-1}(y)) \in \mathbf{R}_{-\infty,a}, \tag{3.1.3(iv)}$$

respectively the following maps:

$„\circ” : \mathbf{R}_{-\infty,m} \times \mathbf{R}_{-\infty,m} \to \mathbf{R}_{-\infty,m}$ and $„*” : \mathbf{R}_{-\infty,m} \times \mathbf{R}_{-\infty,m} \to \mathbf{R}_{-\infty,m}$,

defined by: for every $x, y \in \mathbf{R}_{-\infty,m}$,

$$x \circ y=v(v^{-1}(x)+v^{-1}(y)) \in \mathbf{R}_{-\infty,m}, \tag{3.1.3(v)}$$

respectively, for every $x, y \in \mathbf{R}_{-\infty,m}$,

$$x * y=v(v^{-1}(x) \cdot v^{-1}(y)) \in \mathbf{R}_{-\infty,m}, \tag{3.1.3(vi)}$$

such that $(\mathbf{R}_{-\infty,a},\circ,\bullet)$, respectively $(\mathbf{R}_{-\infty,m},\circ,*)$ to be (commutative) fields, each of them isomorphic to the field $(\mathbf{R},+,\cdot)$, such that the following diagram of isomorphic fields is commutative:

$$\begin{tikzcd}
(\mathbf{R},+,\cdot) \arrow[r, "u"] \arrow[d, "v"'] & (\mathbf{R}_{-\infty,a},\circ,\bullet) \arrow[dl, "f_4=v\circ u^{-1}"] \\
(\mathbf{R}_{-\infty,m},\circ,*). &
\end{tikzcd} \tag{3.4.1}$$

As in the proof of the Corollary 2.2, now, we consider the function:

$$f_4=v \circ u^{-1} : \mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,m}, \tag{3.4.2}$$

which is an isomorphism of fields (being composed of two such functions). Therefore, through the function f_4, the fields $(\mathbf{R}_{-\infty,a},\circ,\bullet)$ and $(\mathbf{R}_{-\infty,m},\circ,*)$ are isomorphic and the diagram (3.4.1) is commutative. Concrete, in our case,

$u^{-1} : \mathbf{R}_{-\infty,a} \to \mathbf{R}$

is defined by: for every $x, y \in \mathbf{R}_{-\infty,a}$,

$$u^{-1}(x)=-\ln(a-x) =\ln \frac{1}{a-x}, \tag{3.1.2}$$

It follows that, the function f_4, given by the equality (3.4.2), becomes: for every $x \in \mathbf{R}_{-\infty,a}$,

$$f_4(x)=(v \circ u^{-1})(x) =m-(a-x). \tag{3.4.3}$$

From the equalities (3.4.3), as we expected, it follows that the function f_4 is bijective – being a first degree function. It is easily verified that the function f_4 is a fields morphism. □

The following remark are required here:

Remark 3.4.2: *The functions u and v from Application 3.4.1 are increasing. Observe that we can also consider descending functions, that is of the type of (3.1.1′); the conclusion of Application 3.4.1 remaining the same.* □

The inductive approach of this isomorphism of fields involves solving the problem:

Problem 3.4.3 *We consider* $a, m \in \mathbf{R}$ *and the following two laws of composition*:

$„\circ” : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *and* $„\bullet” : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$,

respectively

$„\circ” : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *and* $„*” : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$,

defined by: for every $(x,y) \in \mathbf{R} \times \mathbf{R}$,

$x \circ y=a-a^2+a(x+y)-xy$ *and* $x \bullet y=a-(a-x)^{-\ln(a-y)}$,

respectively

$x \circ y=m-m^2+m(x+y)-xy$ *and* $x * y=m-(m-x)^{-\ln(m-y)}$.

Prove that:

a) *the maps „○" and „•" determines on the set*:

$$\boldsymbol{R}_{-\infty,a}=(-\infty,a)$$

a structure of commutative field and that the field $(\boldsymbol{R}_{-\infty,a},\circ,\bullet)$ *is isomorphic to the field* $(\boldsymbol{R},+\cdot)$;

b) *the maps „∘" and „∗" determines on the set*:

$$\boldsymbol{R}_{-\infty,a}=(-\infty,m)$$

a structure of commutative field and that the field $(\boldsymbol{R}_{-\infty,m},\circ,*)$ *is isomorphic to the field* $(\boldsymbol{R},+\cdot)$;

c) *the fields* $(\boldsymbol{R}_{-\infty,a},\circ,\bullet)$ *and* $(\boldsymbol{R}_{-\infty,m},\circ,*)$ *are isomorphic*.

***Hint*:** As with Problem 3.1.3 it shows that the statements in points a) and b) hold. To determine the isomorphism required in point c) we consider the function:

$$f_4 : \mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,m},$$

defined by: for every $x\in\mathbf{R}_{-\infty,a}$,

$$f_4(x)=m-a+x. \qquad (3.4.3)$$

As well as proving the Application 3.4.1, it shows that this function is the sought isomorphism. □

3.5 Fields of the form $(\mathbf{R}_{-\infty,a},\circ,\bullet)$ isopmorphic to the fields of the form $(\mathbf{R}_{b,c},\circ,*)$ (but also to $(\mathbf{R},+,\cdot)$)

***Application 3.5.1*:** *For every a, b, c* $\in\boldsymbol{R}$, *cu* $b<c$, *there are two laws of internal composition on the set*:

$$\boldsymbol{R}_{-\infty,a}=(-\infty,a)$$

and there are two laws of internal composition on the set:

$$\boldsymbol{R}_{b,c}=(b,c),$$

which to determine on these intervals a structure of (commutative) field and the two fields to be isomorphic between them, but also to the field $(\boldsymbol{R},+,\cdot)$.

***Hint*:** Observe that this application is a particular case of Corollary 2.2, for:

$$\mathbf{R}_{-\infty,a}=(-\infty,a) \quad \text{and} \quad \mathbf{R}_{b,c}=(b,c).$$

Therefore, according to Application 3.1.1 and according to Application 3.2.1, there are the bijective functions:

$$u : \mathbf{R} \to \mathbf{R}_{-\infty,a} \quad \text{and} \quad v : \mathbf{R} \to \mathbf{R}_{b,c},$$

defined by: for every $x\in\mathbf{R}$,

$$u(x)=a-e^{-x}, \qquad (3.1.1)$$

respectively,

$$v(x)=\frac{be^{x}+c}{1+e^{x}}, \qquad (3.2.1)$$

which determines the following maps:

$$„\circ" : \mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,a} \quad \text{and} \quad „\bullet" : \mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,a},$$

defined by: for every $x, y\in\mathbf{R}_{-\infty,a}$,

$$x\circ y=u(u^{-1}(x)+u^{-1}(y))\in\mathbf{R}_{-\infty,a}, \qquad (3.1.3'')$$

$$x\bullet y=u(u^{-1}(x)\cdot u^{-1}(y))\in\mathbf{R}_{-\infty,a}, \qquad (3.1.3^{(iv)})$$

respectively the following maps:

$$„\circ" : \mathbf{R}_{-\infty,m}\times\mathbf{R}_{-\infty,m} \to \mathbf{R}_{-\infty,m} \quad \text{and} \quad „*" : \mathbf{R}_{-\infty,m}\times\mathbf{R}_{-\infty,m} \to \mathbf{R}_{-\infty,m},$$

defined by: for every $x, y\in\mathbf{R}_{-\infty,m}$,

$$x\circ y=v(v^{-1}(x)+v^{-1}(y))\in\mathbf{R}_{-\infty,m}, \qquad (3.2.3')$$

$$x*y=v(v^{-1}(x)\cdot v^{-1}(y))\in\mathbf{R}_{-\infty,m}, \qquad (3.2.4')$$

such that $(\mathbf{R}_{-\infty,a},\circ,\bullet)$, respectively $(\mathbf{R}_{b,c},\circ,*)$ to be (commutative) fields, each of them isomorphic to the field $(\mathbf{R},+,\cdot)$, such that the following diagram of isomorphic fields is commutative:

$$\begin{tikzcd}
(\mathbf{R},+,\cdot) \arrow[r, "u"] \arrow[d, "v"'] & (\mathbf{R}_{-\infty,a},\circ,\bullet) \arrow[dl, "f_5=v\circ u^{-1}"] \\
(\mathbf{R}_{b,c},\circ,*) &
\end{tikzcd} \qquad (3.5.1)$$

As in the proof of the Corollary 2.2, now, we consider function:

$$f_5=v\circ u^{-1} : \mathbf{R}_{-\infty,a} \to \mathbf{R}_{b,c}, \qquad (3.5.2)$$

which is an isomorphism of fields (being composed of two such functions). Therefore, through the function f_5, the fields ($\mathbf{R}_{-\infty,a}$,○,•) and ($\mathbf{R}_{b,c}$,∘,∗) are isomorphic and the diagram (3.5.1) is commutative. Concrete, in our case,

$u^{-1} : \mathbf{R}_{-\infty,a} \to \mathbf{R}$

is defined by: for every x, y∈$\mathbf{R}_{-\infty,a}$,

$$u^{-1}(x)=-\ln(a-x)$$
$$=\ln\frac{1}{a-x}, \qquad (3.1.2)$$

It follows that, the function f_5, given by the equality (3.5.2), becomes: for every x∈$\mathbf{R}_{-\infty,a}$,

$$f_5(x)=(v\circ u^{-1})(x)$$
$$=\frac{ac+b-cx}{a+1-x}. \qquad (3.5.3)$$

It is easily verified that the function f_5 is a fields isomorphism. □

The following remark are required here:

Remark 3.5.2: *In Application 3.5.1 the function u is increasing, and the function v is decreasing. Observe that we can also consider case in which the function u is decreasing, that is of the type of (3.1.1′), and the function v is increasing, that is of the type of (3.2.1′); the conclusion of Application 3.5.1 remaining the same.* □

The inductive approach of this isomorphism of fields involves solving the problem:

Problem 3.5.3 *We consider a, b, c∈**R**, cu b<c and the following two laws of composition:*

„○" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ and „•" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

respectively

„∘" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ and „∗" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

*defined by: for every (x,y)∈**R**×**R**,*

$$x\circ y=a-a^2+a(x+y)-xy \quad \text{and} \quad x\bullet y=a-(a-x)^{-\ln(a-y)},$$

respectively

$$x\circ y=\frac{xy(b+c)-2bc(x+y)+bc(b+c)}{2xy-(b+c)(x+y)+b^2+c^2} \quad \text{and} \quad x*y==\frac{b\left(\frac{c-x}{x-b}\right)^{\ln\left(\frac{c-y}{y-b}\right)}+c}{1+\left(\frac{c-x}{x-b}\right)^{\ln\left(\frac{c-y}{y-b}\right)}}.$$

Prove that:

a) *the maps „○" and „•"determines on the set:*

$$\mathbf{R}_{-\infty,a}=(-\infty,a)$$

*a structure of commutative field and that the field ($\mathbf{R}_{-\infty,a}$,○,•) is isomorphic to the field (**R**,+·);*

b) *the maps „∘" and „∗" determines on the set:*

$$\mathbf{R}_{b,c}=(b,c)$$

*a structure of commutative field and that the field ($\mathbf{R}_{b,c}$,∘,∗) is isomorphic to the field (**R**,+·);*

c) *the fields ($\mathbf{R}_{-\infty,a}$,○,•) and ($\mathbf{R}_{b,c}$,∘,∗) are isomorphic.*

Solution: As with Problem 3.1.3, respectively Problem 3.2.3 it shows that the statements in points a), repectively b) hold. To determine the isomorphism required in point c) we consider the function:

$f_5 : \mathbf{R}_{-\infty,a} \to \mathbf{R}_{b,,c}$,

defined by: for every x∈$\mathbf{R}_{-\infty,a}$,

$$f_5(x)=\frac{ac+b-cx}{a+1-x}. \qquad (3.5.3)$$

As well as proving the Application 3.5.1, it shows that this function is the sought isomorphism. □

3.6 Fields of the form ($\mathbf{R}_{b,c}$,○,•) isopmorphic to the fields of the form ($\mathbf{R}_{n,p}$,∘,∗) (but also to (R,+,·))

Application 3.6.1: *For every b, c, n, p∈**R**, cu b<c and n<p, there are two laws of internal composition on the set:*

$$\mathbf{R}_{b,c}=(b,c)$$

and there are two laws of internal composition on the set:

$$\mathbf{R}_{n,p}=(n,p),$$

*which to determine on these intervals a structure of (commutative) field and the two fields to be isomorphic between them, but also to the field (**R**,+,·).*

Hint: Observe that this application is a particular case of Corollary 2.2, for:

$\mathbf{R}_{b,c}=(b,c)$ and $\mathbf{R}_{n,p}=(n,p)$.

Therefore, according to Application 3.2.1, there are the bijective functions:

$u:\mathbf{R}\to\mathbf{R}_{b,c}$ and $v:\mathbf{R}\to\mathbf{R}_{n,p}$,

defined by: for every $x\in\mathbf{R}$,

$$u(x)=\frac{be^x+c}{1+e^x}, \qquad (3.2.1)$$

respectively,

$$v(x)=\frac{ne^x+p}{1+e^x}, \qquad (3.2.1'')$$

which determines the following maps:

„$\circ$” : $\mathbf{R}_{b,c}\times\mathbf{R}_{b,c}\to\mathbf{R}_{b,c}$ and „$\bullet$” : $\mathbf{R}_{b,c}\times\mathbf{R}_{b,c}\to\mathbf{R}_{b,c}$,

defined by: for every $x, y\in\mathbf{R}_{b,c}$,

$$x\circ y=u(u^{-1}(x)+u^{-1}(y))\in\mathbf{R}_{b,c}, \qquad (3.2.3')$$

$$x\bullet y=u(u^{-1}(x)\cdot u^{-1}(y))\in\mathbf{R}_{b,c}, \qquad (3.2.4')$$

respectively the following maps:

„$\circ$” : $\mathbf{R}_{n,p}\times\mathbf{R}_{n,p}\to\mathbf{R}_{n,p}$ and „$*$” : $\mathbf{R}_{n,p}\times\mathbf{R}_{n,p}\to\mathbf{R}_{n,p}$,

defined by: for every $x, y\in\mathbf{R}_{n,p}$,

$$x\circ y=v(v^{-1}(x)+v^{-1}(y))\in\mathbf{R}_{n,p}, \qquad 3.2.3'')$$

$$x*y=v(v^{-1}(x)\cdot v^{-1}(y))\in\mathbf{R}_{n,p}, \qquad (3.2.4'')$$

such that $(\mathbf{R}_{b,c},\circ,\bullet)$, respectively $(\mathbf{R}_{n,p},\circ,*)$ to be (commutative) fields, each of them isomorphic to the field $(\mathbf{R},+,\cdot)$, such that the following diagram of isomorphic fields is commutative:

$$\begin{tikzcd}
(\mathbf{R},+,\cdot) \arrow[r, "u"] \arrow[d, "v"'] & (\mathbf{R}_{b,c},\circ,\bullet) \arrow[dl, "f_6=v\circ u^{-1}"] \\
(\mathbf{R}_{n,p},\circ,*) &
\end{tikzcd} \qquad (3.6.1)$$

As in the proof of the Corollary 2.2, now, we consider function:

$$f_6=v\circ u^{-1}:\mathbf{R}_{b,c}\to\mathbf{R}_{n,p}, \qquad (3.6.2)$$

which is an isomorphism of fields (being composed of two such functions). Therefore, through the function f_6, the fields $(\mathbf{R}_{b,c},\circ,\bullet)$ and $(\mathbf{R}_{n,p},\circ,*)$ are isomorphic and the diagram (3.6.1) is commutative. Concrete, in our case,

$u^{-1}:\mathbf{R}_{b,c}\to\mathbf{R}$

and is defined by: for every $x\in\mathbf{R}_{b,c}$,

$$u^{-1}(x)=\ln\frac{c-x}{x-b}. \qquad (3.2.2)$$

It follows that, the function f_6, given by the equality (3.6.2), becomes: for every $x\in\mathbf{R}_{b,c}$,

$$f_6(x)=(v\circ u^{-1})(x)=\frac{(p-n)x+nc-bp}{c-b}. \qquad (3.6.3)$$

It is easily verified that the function f_6 is a fields isomorphism. □

The following remark are required here:

Remark 3.6.2: *In Application 3.6.1 the functions u, respectively v are decreasing. Observe that we can also consider case in which one of them or even both to be increasing, that is of the type of (3.2.1'); the conclusion of Application 3.6.1 remaining the same.* □

The inductive approach of this isomorphism of fields involves solving the problem:

Problem 3.6.3 *We consider $b, c, n, p\in\mathbf{R}$, cu $b<c$ and $n<p$ and the following two laws of composition:*

„$\circ$” : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ and „$\bullet$” : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

respectively

„$\circ$” : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ and „$*$” : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

defined by: for every $(x,y)\in\mathbf{R}\times\mathbf{R}$,

$$x\circ y=\frac{xy(b+c)-2bc(x+y)+bc(b+c)}{2xy-(b+c)(x+y)+b^2+c^2} \quad \text{and} \quad x\bullet y=\frac{b\left(\frac{c-x}{x-b}\right)^{ln\left(\frac{c-y}{y-b}\right)}+c}{1+\left(\frac{c-x}{x-b}\right)^{ln\left(\frac{c-y}{y-b}\right)}},$$

respectively

$$x\circ y=\frac{xy(n+p)-2np(x+y)+np(n+p)}{2xy-(n+p)(x+y)+n^2+p^2} \quad \text{and} \quad x* y=\frac{n\left(\frac{p-x}{x-n}\right)^{ln\left(\frac{p-y}{y-n}\right)}+p}{1+\left(\frac{p-x}{x-n}\right)^{ln\left(\frac{p-y}{y-n}\right)}}.$$

Prove that:

a) *the maps „○" and „•"determines on the set*:

$\boldsymbol{R}_{b,c}=(b,c)$

a structure of commutative field and that the field ($\boldsymbol{R}_{b,c}$, ○, •) *is isomorphic to the field* ($\boldsymbol{R}$,+·);

b) *the maps „∘" and „*" determines on the set*:

$\boldsymbol{R}_{n,p}=(n,p)$

a structure of commutative field and that the field ($\boldsymbol{R}_{n,p}$,∘, *) *is isomorphic to the field* ($\boldsymbol{R}$,+·);

c) *the fields* ($\boldsymbol{R}_{b,c}$, ○, •) *and* ($\boldsymbol{R}_{n,p}$,∘, *) *are isomorphic*.

Hint: As with Problem 3.2.3 it shows that the statements in points a), repectively b) hold. To determine the isomorphism required in point c) we consider the function:

$f_6 : \mathbf{R}_{b,c} \to \mathbf{R}_{n,p}$,

defined by: for every $x\in\mathbf{R}_{b,c}$,

$$f_6(x)=\frac{(p-n)x+nc-bp}{c-b}. \tag{3.6.3}$$

As well as proving the Application 3.6.1, it shows that this function is the sought isomorphism. □

3.7 Fields of the form ($\mathbf{R}_{b,c}$,○,●) isopmorphic to the fields of the form ($\mathbf{R}_{d,+\infty}$,∘,*) (but also to the field (R,+,·))

Application 3.7.1: *For every b, c, d*∈$\boldsymbol{R}$, *cu b<c, there are two laws of internal composition on the set*:

$\boldsymbol{R}_{b,c}=(b,c)$

and there are two laws of internal composition on the set:

$\boldsymbol{R}_{d,+\infty}=(d,+\infty)$,

which to determine on these intervals a structure of (commutative) field and the two fields to be isomorphic between them, but also to the field ($\boldsymbol{R}$,+,·).

Proof: Observe that this application is a particular case of Corollary 2.2, for:

$\mathbf{R}_{b,c}=(b,c)$ and $\mathbf{R}_{d,+\infty}=(d,+\infty)$.

Therefore, according to Application 3.2.1 and according to Application 3.3.1, there are the bijective functions:

$u : \mathbf{R} \to \mathbf{R}_{b,c}$ and $v : \mathbf{R} \to \mathbf{R}_{d,+\infty}$,

defined by: for every $x\in\mathbf{R}$,

$$u(x)=\frac{be^x+c}{1+e^x}, \tag{3.2.1}$$

respectively,

$$v(x)=d+e^x, \tag{3.3.1}$$

which determines the following maps:

„○" : $\mathbf{R}_{b,c}\times\mathbf{R}_{b,c} \to \mathbf{R}_{b,c}$ and „●" : $\mathbf{R}_{b,c}\times\mathbf{R}_{b,c} \to \mathbf{R}_{b,c}$,

defined by: for every x, $y\in\mathbf{R}_{b,c}$,

$$x\circ y=u(u^{-1}(x)+u^{-1}(y))\in\mathbf{R}_{b,c}, \tag{3.2.3′}$$

$$x\bullet y=u(u^{-1}(x)\cdot u^{-1}(y))\in\mathbf{R}_{b,c}, \tag{3.2.4′}$$

respectively the following maps:

„$\circ$" : $\mathbf{R}_{d,+\infty}\times\mathbf{R}_{d,+\infty}\to\mathbf{R}_{d,+\infty}$ and „$*$" : $\mathbf{R}_{d,+\infty}\times\mathbf{R}_{d,+\infty}\to\mathbf{R}_{d,+\infty}$,

defined by: for every x, $y\in\mathbf{R}_{d,+\infty}$,

$$x\circ y=v(v^{-1}(x)+v^{-1}(y))\in\mathbf{R}_{d,+\infty}, \tag{3.3.3′}$$

$$x*y=v(v^{-1}(x)\cdot v^{-1}(y))\in\mathbf{R}_{d,+\infty}, \tag{3.3.4′}$$

such that $(\mathbf{R}_{b,c},\circ,\bullet)$, respectively $(\mathbf{R}_{n,p},\circ,*)$ to be (commutative) fields, each of them isomorphic to the field $(\mathbf{R},+,\cdot)$, such that the following diagram of isomorphic fields is commutative:

$$\begin{tikzcd} (\mathbf{R},+,\cdot) \arrow[r, "u"] \arrow[d, "v"'] & (\mathbf{R}_{b,c},\circ,\bullet) \arrow[dl, "f_7=v\circ u^{-1}"] \\ (\mathbf{R}_{d,+\infty},\circ,*) & \end{tikzcd} \tag{3.7.1}$$

As in the proof of the Corollary 2.2, now, we consider function:

$$f_7=v\circ u^{-1} : \mathbf{R}_{b,c}\to\mathbf{R}_{d,+\infty}, \tag{3.7.2}$$

which is an isomorphism of fields (being composed of two such functions). Therefore, through the function f_7, the fields $(\mathbf{R}_{b,c},\circ,\bullet)$ and $(\mathbf{R}_{d,+\infty},\circ,*)$ are isomorphic and the diagram (3.7.1) is commutative. Concrete, in our case,

$$u^{-1} : \mathbf{R}_{b,c}\to\mathbf{R}$$

and is defined by: for every $x\in\mathbf{R}_{b,c}$,

$$u^{-1}(x)=\ln\frac{c-x}{x-b}. \tag{3.2.2}$$

It follows that, the function f_7, given by the equality (3.7.2), becomes: for every $x\in\mathbf{R}_{b,c}$,

$$f_7(x)=(v\circ u^{-1})(x)$$

$$=\frac{x(d-1)+c-bd}{x-b}. \tag{3.7.3}$$

It is easily verified that the function f_5 is a fields isomorphism. □

The following remark are required here:

Remark 3.7.2: *In Application 3.7.1 the function u is decreasing, and the function v is increasing. Observe that we can also consider case in which the function u is increasing, that is of the type of (3.2.1′), and the function v is decreasing, that is of the type of (3.3.1′); the conclusion of Application 3.7.1 remaining the same.* □

The inductive approach of this isomorphism of fields involves solving the problem:

Problem 3.7.3 *We consider b, c, $d\in\mathbf{R}$, cu $b<c$ and the following two laws of composition:*

„$\circ$" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *and* „$\bullet$" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

respectively

„$\circ$" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *and* „$*$" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

defined by: for every $(x,y)\in\mathbf{R}\times\mathbf{R}$,

$$x\circ y=\frac{xy(b+c)-2bc(x+y)+bc(b+c)}{2xy-(b+c)(x+y)+b^2+c^2} \quad \text{and} \quad x\bullet y=\frac{b\left(\frac{c-x}{x-b}\right)^{\ln\left(\frac{c-y}{y-b}\right)}+c}{1+\left(\frac{c-x}{x-b}\right)^{\ln\left(\frac{c-y}{y-b}\right)}},$$

respectively

$$x\circ y=d+(x-d)(y-d) \quad \text{and} \quad x*y=d+(x-d)^{\ln(y-d)}.$$

Prove that:

a) *the maps „$\circ$" and „$\bullet$" determines on the set:*

$$\mathbf{R}_{b,c}=(b,c)$$

a structure of commutative field and that the field $(\mathbf{R}_{b,c},\circ,\bullet)$ is isomorphic to the field $(\mathbf{R},+\cdot)$;

b) *the maps „$\circ$" and „$*$" determines on the set:*

$$\mathbf{R}_{d,+\infty}=(d,+\infty)$$

a structure of commutative field and that the field $(\mathbf{R}_{d,+\infty},\circ,*)$ is isomorphic to the field $(\mathbf{R},+\cdot)$;

***c)** the fields* $(\mathbf{R}_{b,c},\circ,\bullet)$ *and* $(\mathbf{R}_{d,+\infty},\circ,*)$ *are isomorphic*.

***Hint*:** As with Problem 3.2.3, respectively Problem 3.3.3 it shows that the statements in points a), repectively b) hold. To determine the isomorphism required in point c) we consider the function:

$f_7 : \mathbf{R}_{b,c} \to \mathbf{R}_{d,+\infty}$,

defined by: for every $x\in\mathbf{R}_{b,c}$,

$$f_7(x)=\frac{x(d-1)+c-bd}{x-b}. \qquad (3.7.3)$$

As well as proving the Application 3.7.1, it shows that this function is the sought isomorphism. □

3.8 Fields of the form $(\mathbf{R}_{d,+\infty},\circ,\bullet)$ isomorphic between them (but also to the field $(\mathbf{R},+,\cdot)$)

***Application 3.8.1*:** *For every d, q* $\in\mathbf{R}$, *there are two laws of internal composition on the set*:

$\mathbf{R}_{d,+\infty}=(d,+\infty)$

and there are two laws of internal composition on the set:

$\mathbf{R}_{q,+\infty}=(q,+\infty)$,

which to determine on these intervals a structure of (commutative) field and the two fields to be isomorphic between them, but also to the field $(\mathbf{R},+,\cdot)$.

***Hint*:** Observe that this application is a particular case of Corollary 2.2, for:

$\mathbf{R}_{d,+\infty}=(d,+\infty)$ and $\mathbf{R}_{q,+\infty}=(q,+\infty)$.

Therefore, according to Application 3.3.1, there are the bijective functions:

$u : \mathbf{R} \to \mathbf{R}_{d,+\infty}$ and $v : \mathbf{R} \to \mathbf{R}_{q,+\infty}$,

defined by: for every $x\in\mathbf{R}$,

$$u(x)=d+e^x, \qquad (3.3.1)$$

respectively,

$$v(x)=q+e^x, \qquad (3.3.1'')$$

which determines the following maps:

„$\circ$” : $\mathbf{R}_{d,+\infty}\times\mathbf{R}_{d,+\infty} \to \mathbf{R}_{d,+\infty}$ and „$\bullet$” : $\mathbf{R}_{d,+\infty}\times\mathbf{R}_{d,+\infty} \to \mathbf{R}_{d,+\infty}$,

defined by: for every x, $y\in\mathbf{R}_{d,+\infty}$,

$$x\circ y=u(u^{-1}(x)+u^{-1}(y))\in\mathbf{R}_{d,+\infty}, \qquad (3.3.3')$$

$$x\bullet y=u(u^{-1}(x)\cdot u^{-1}(y))\in\mathbf{R}_{d,+\infty}, \qquad (3.3.4')$$

respectively the following maps:

„$\circ$” : $\mathbf{R}_{q,+\infty}\times\mathbf{R}_{q,+\infty} \to \mathbf{R}_{q,+\infty}$ and „$*$” : $\mathbf{R}_{q,+\infty}\times\mathbf{R}_{q,+\infty} \to \mathbf{R}_{q,+\infty}$,

defined by: for every x, $y\in\mathbf{R}_{q,+\infty}$,

$$x\circ y=v(v^{-1}(x)+v^{-1}(y))\in\mathbf{R}_{q,+\infty}, \qquad (3.3.3'')$$

$$x*y=v(v^{-1}(x)\cdot v^{-1}(y))\in\mathbf{R}_{q,+\infty}, \qquad (3.3.4'')$$

such that $(\mathbf{R}_{d,+\infty},\circ,\bullet)$, respectively $(\mathbf{R}_{q,+\infty},\circ,*)$ to be (commutative) fields, each of them isomorphic to the field $(\mathbf{R},+,\cdot)$, such that the following diagram of isomorphic fields is commutative:

$$\begin{tikzcd}
(\mathbf{R},+,\cdot) \arrow[r, "u"] \arrow[d, "v"'] & (\mathbf{R}_{d,+\infty},\circ,\bullet) \arrow[dl, "f_8=v\circ u^{-1}."] \\
(\mathbf{R}_{q,+\infty},\circ,*) &
\end{tikzcd} \qquad (3.8.1)$$

As in the proof of the Corollary 2.2, now, we consider function:

$$f_8=v\circ u^{-1} : \mathbf{R}_{d,+\infty} \to \mathbf{R}_{q,+\infty}, \qquad 3.8.2)$$

which is an isomorphism of fields (being composed of two such functions). Therefore, through the function f_5, the fields $(\mathbf{R}_{d,+\infty},\circ,\bullet)$ and $(\mathbf{R}_{q,+\infty},\circ,*)$ are isomorphic and the diagram (3.8.1) is commutative. Concrete, in our case,

$u^{-1} : \mathbf{R}_{d,+\infty} \to \mathbf{R}$

and is defined by: for every $x\in\mathbf{R}_{d,+\infty}$,

$$u^{-1}(x)=\ln(x-d). \qquad (3.3.2)$$

It follows that, the function f_8, given by the equality (3.8.2), becomes: for every $x\in\mathbf{R}_{-\infty,a}$,

$f_8(x)=(v\circ u^{-1})(x)$

$=q-d+x.$ (3.8.3)

It is easily verified that the function f_8 is a fields isomorphism. □

The following remark are required here:

Remark 3.8.2: *In Application 3.8.1 the functions u and v are increasing. Observe that we can also consider case in which one of them or even both to be decreasing, that is of the type of (3.3.1′); the conclusion of Application 3.8.1 remaining the same.* □

The inductive approach of this isomorphism of fields involves solving the problem:

Problem 3.8.3 *We consider $d, q\in\mathbf{R}$, and the following two laws of composition:*

$„\circ" : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ *and* $„\bullet" : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$,

respectively

$„\circ" : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ *and* $„*" : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$,

defined by: for every $(x,y)\in\mathbf{R}\times\mathbf{R}$,

$x\circ y=d+(x-d)(y-d)$ *and* $x\bullet y=d+(x-d)^{\ln(y-d)}$,

respectively

$x\circ y=q+(x-q)(y-q)$ *and* $x*y=q+(x-q)^{\ln(y-q)}$.

Prove that:

a) *the maps „○" and „•" determines on the set:*

$\mathbf{R}_{d,+\infty}=(d,+\infty)$

a structure of commutative field and that the field $(\mathbf{R}_{d,+\infty},\circ,\bullet)$ is isomorphic to the field $(\mathbf{R},+,\cdot)$;

b) *the maps „∘" and „*" determines on the set:*

$\mathbf{R}_{q,+\infty}=(q,+\infty)$

a structure of commutative field and that the field $(\mathbf{R}_{q,+\infty},\circ,*)$ is isomorphic to the field $(\mathbf{R},+,\cdot)$;

c) *the fields $(\mathbf{R}_{d,+\infty},\circ,\bullet)$ and $(\mathbf{R}_{q,+\infty},\circ,*)$ are isomorphic.*

***Hint*:** As with Problem 3.3.3 it shows that the statements in points a), repectively b) hold. To determine the isomorphism required in point c) we consider the function:

$f_8 : \mathbf{R}_{d,+\infty} \to \mathbf{R}_{q,+\infty}$,

defined by: for every $x\in\mathbf{R}_{d,+\infty}$,

$f_8(x)=q-d+x.$ (3.8.3)

As well as proving the Application 3.8.1, it shows that this function is the sought isomorphism. □

3.9 Fields of the form $(\mathbf{R}_{-\infty,a},\circ,\bullet)$ isopmorphic to the fields of the form $(\mathbf{R}_{d,+\infty},\circ,*)$ (but also to $(\mathbf{R},+,\cdot)$)

***Application 3.9.1*:** *For every $a, d\in\mathbf{R}$, there are two laws of internal composition on the set:*

$\mathbf{R}_{-\infty,a}=(-\infty,a)$

and there are two laws of internal composition on the set:

$\mathbf{R}_{d,+\infty}=(d,+\infty)$,

which to determine on these intervals a structure of (commutative) field and the two fields to be isomorphic between them, but also to the field $(\mathbf{R},+,\cdot)$.

***Hint*:** Observe that this application is a particular case of Corollary 2.2, for:

$\mathbf{R}_{-\infty,a}=(-\infty,a)$ and $\mathbf{R}_{d,+\infty}=(d,+\infty)$.

Therefore, according to Application 3.1.1 and according to Application 3.3.1, there are the bijective functions:

$u : \mathbf{R} \to \mathbf{R}_{-\infty,a}$ and $v : \mathbf{R} \to \mathbf{R}_{d,+\infty}$,

defined by: for every $x\in\mathbf{R}$,

$u(x)=a-e^{-x}$, (3.1.1)

respectively,

$v(x)=d+e^{x}$, (3.3.1)

which determines the following maps:

$„\circ" : \mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,a}$ and $„\bullet" : \mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a} \to \mathbf{R}_{-\infty,a}$,

defined by: for every $x, y\in\mathbf{R}_{-\infty,a}$,

$x\circ y=u(u^{-1}(x)+u^{-1}(y))\in\mathbf{R}_{-\infty,a}$, (3.1.3)

$x\bullet y=u(u^{-1}(x)\cdot u^{-1}(y))\in\mathbf{R}_{-\infty,a}$, (3.1.3′)

respectively the following maps:

$„\circ" : \mathbf{R}_{d,+\infty}\times\mathbf{R}_{d,+\infty} \to \mathbf{R}_{d,+\infty}$ and $„*" : \mathbf{R}_{d,+\infty}\times\mathbf{R}_{d,+\infty} \to \mathbf{R}_{d,+\infty}$,

defined by: for every $x, y\in\mathbf{R}_{d,+\infty}$,

$x\circ y=v(v^{-1}(x)+v^{-1}(y))\in\mathbf{R}_{d,+\infty}$, (3.3.3)

$$x*y=v(v^{-1}(x)\cdot v^{-1}(y))\in \mathbf{R}_{d,+\infty}, \tag{3.3.4}$$

such that ($\mathbf{R}_{-\infty,a}$,○,•), respectively ($\mathbf{R}_{d,+\infty}$,∘,∗) to be (commutative) fields, each of them isomorphic to the field ($\mathbf{R}$,+,·), such that the following diagram of isomorphic fields is commutative:

$$\begin{tikzcd}
(\mathbf{R},+,\cdot) \arrow[r, "u"] \arrow[d, "v"'] & (\mathbf{R}_{-\infty,a},\circ,\bullet) \arrow[dl, "f_9=v\circ u^{-1}"] \\
(\mathbf{R}_{d,+\infty},\circ,*) &
\end{tikzcd} \tag{3.9.1}$$

As in the proof of the Corollary 2.2, now, we consider function:

$$f_9=v\circ u^{-1} : \mathbf{R}_{-\infty,a} \to \mathbf{R}_{d,+\infty}, \tag{3.9.2}$$

which is an isomorphism of fields (being composed of two such functions). Therefore, through the function f_9, the fields ($\mathbf{R}_{-\infty,a}$,○,•) and ($\mathbf{R}_{d,+\infty}$,∘,∗) are isomorphic and the diagram (3.9.1) is commutative. Concrete, in our case,

$$u^{-1} : \mathbf{R}_{-\infty,a} \to \mathbf{R}$$

and is defined by: for every $x\in\mathbf{R}_{-\infty,a}$,

$$u^{-1}(x)=\ln\frac{1}{a-x}. \tag{3.1.2}$$

It follows that, the function f_9, given by the equality (3.9.2), becomes: for every $x\in\mathbf{R}_{-\infty,a}$,

$$f_9(x)=(v\circ u^{-1})(x)$$
$$=\frac{ad+1-dx}{a-x}. \tag{3.9.3}$$

It is easily verified that the function f_9 is a fields isomorphism. □

The following remark are required here:

Remark 3.9.2: *In Application 3.9.1 the functions u and v are increasing. Observe that we can also consider case in which one of them or even both to be decreasing, that is u to be of the type of (3.1.1′) and v to be of the type of (3.3.1′); the conclusion of Application 3.9.1 remaining the same.* □

The inductive approach of this isomorphism of fields involves solving the problem:

Problem 3.9.3 *We consider $a, d\in\mathbf{R}$, and the following two laws of composition*:

„○” : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *and* „●” : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

respectively

„∘” : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *and* „∗” : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

defined by: for every $(x,y)\in\mathbf{R}\times\mathbf{R}$,

$x\bigcirc y=a-a^2+a(x+y)-xy$ *and* $x\bullet y=a-(a-x)^{-\ln(a-y)}$,

respectively

$x\circ y=d+(x-d)(y-d)$ *and* $x*y=d+(x-d)^{\ln(y-d)}$.

Prove that:

a) *the maps „○” and „•”determines on the set*:

$$\mathbf{R}_{-\infty,a}=(-\infty,a)$$

a structure of commutative field and that the field ($\mathbf{R}_{-\infty,a}$, ○, •) is isomorphic to the field ($\mathbf{R}$,+·);

b) *the maps „∘” and „∗” determines on the set*:

$$\mathbf{R}_{d,+\infty}=(d,+\infty)$$

a structure of commutative field and that the field ($\mathbf{R}_{d,+\infty}$∘, ∗) is isomorphic to the field ($\mathbf{R}$,+·);

c) *the fields ($\mathbf{R}_{-\infty,a}$, ○, •) and ($\mathbf{R}_{d,+\infty}$, ∘, ∗) are isomorphic.*

Hint: As with Problem 3.1.3, respectively Problem 3.3.3 it shows that the statements in points a), repectively b) hold. To determine the isomorphism required in point c) we consider the function:

$$f_9 : \mathbf{R}_{-\infty,a} \to \mathbf{R}_{d,+\infty},$$

defined by: for every $x\in\mathbf{R}_{-\infty,a}$,

$$f_9(x)=\frac{ad+1-dx}{a-x}. \tag{3.9.3}$$

As well as proving the Application 3.9.1, it shows that this function is the sought isomorphism. □

3.10 Fields of the form ($R_{-\infty,a}$,○,●), ($R_{-a,a}$,∘,∗) and ($R_{a,+\infty}$,Δ,⊥) isomorphic between them (but also to the field (R,+,·))

Application 3.10.1: *For every $a\in(0,+\infty)$, the following statements hold*:

a) *there are two laws of internal composition on the set*:

$\boldsymbol{R}_{-\infty,-a}=(-\infty,-a)$,

which to determine on this interval a structure of field (commutative);

b) *there are two laws of internal composition on the set*:

$\boldsymbol{R}_{-a,a}=(-a,a)$,

which to determine on this interval a structure of field (commutative) isomorph to the field;

c) *there are two laws of internal composition on the set*:

$\boldsymbol{R}_{a,+\infty}=(a,+\infty)$,

which to determine on this interval a structure of field (commutative) isomorph to the field;

d) *the three fields defined above are isomorphic between them.*

***Proof*: a)** Retracing the proof of Application 3.1.1 replacing a with -a it shows that the following maps:

„○" : $\mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a}\to\mathbf{R}_{-\infty,a}$ and „●" : $\mathbf{R}_{-\infty,a}\times\mathbf{R}_{-\infty,a}\to\mathbf{R}_{-\infty,a}$,

defined by: for every x, y$\in\mathbf{R}_{-\infty,-a}$,

$$x\circ y=-a-a^2-a(x+y)-xy, \tag{3.1.4″}$$

respectively

$$x\bullet y=-a-(-a-x)^{-\ln(-a-y)}. \tag{3.1.4′}$$

there are two laws of composition on $\mathbf{R}_{-\infty,-a}$, which determines on this set a structure of (commutative) field, isomorphic to the field (**R**,+,·), and the function:

$f:\mathbf{R}\to\mathbf{R}_{-\infty,-a}$,

defined by: for every $x\in\mathbf{R}$,

$$f(x)=-a-e^{-x}, \tag{3.1.1‴}$$

is a field isomorphism from the field (**R**,+,·) to the field ($\mathbf{R}_{-\infty,-a}$,○,●), in which case the inverse of function f, that is, the function:

$f^{-1}:\mathbf{R}_{-\infty,-a}\to\mathbf{R}$,

defined by: for every $x\in\mathbf{R}_{-\infty,-a}$,

$$f^{-1}(x)=\ln\frac{1}{-a-x}, \tag{3.1.2‴}$$

is a field isomorphism from the field ($\mathbf{R}_{-\infty,-a}$,○,●) to the field (**R**,+,·).

b) Observe that, here, we have a particular case of Application 3.2.1 for:

b=-a and c=a.

Therefore, according to the proof of this application, the following maps:

„∘" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ and „∗" : $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

defined by: for every x, y$\in\mathbf{R}_{-a,a}$,

$$x\circ y=\frac{a^2(x+y)}{xy+a^2}, \tag{3.2.3‴}$$

respectively

$$x*y=\frac{a-a\left(\frac{a-x}{x+a}\right)^{\ln\left(\frac{a-y}{y+a}\right)}}{1+\left(\frac{a-x}{x+a}\right)^{\ln\left(\frac{a-y}{y+a}\right)}}, \tag{3.2.4‴}$$

there are two laws of composition on $\mathbf{R}_{-a,a}$, which determines on this set a structure of (commutative) field, isomorphic to the field (**R**,+,·), and the function:

$g:\mathbf{R}\to\mathbf{R}_{-a,a}$,

defined by: for every $x\in\mathbf{R}$,

$$g(x)=\frac{a(1-e^x)}{1+e^x}, \tag{3.2.1‴}$$

is an isomorphism between the two fields, in which case the function:

$g^{-1} : \mathbf{R}_{-a,a} \to \mathbf{R}$,

defined by: for every $x\in\mathbf{R}_{-a,a}$,

$$g^{-1}(x)=\ln\frac{a-x}{a+x}, \tag{3.2.2‴}$$

is a field isomorphism from the field $(\mathbf{R}_{-a,a},\circ,*)$ to the field $(\mathbf{R},+,\cdot)$.

c) Observe that, here, we have a particular case of Application 3.3.1 for:

$d=a$.

Therefore, according to the proof of this application, the following maps:

„Δ” : $\mathbf{R}_{a,+\infty}\times\mathbf{R}_{a,+\infty} \to \mathbf{R}_{a,+\infty}$ and „$\perp$” : $\mathbf{R}_{a,+\infty}\times\mathbf{R}_{a,+\infty} \to \mathbf{R}_{a,+\infty}$,

defined by: for every $x, y\in\mathbf{R}_{a,+\infty}$,

$$x\Delta y=a+a^2-a(x+y)+xy, \tag{3.3.3‴}$$

respectively

$$x\perp y=a+(x-a)^{\ln(y-a)}, \tag{3.3.4‴}$$

there are two laws of composition on $\mathbf{R}_{a,+\infty}$, which determines on this set a structure of (commutative) field, isomorphic to the field $(\mathbf{R},+,\cdot)$, and the function:

$h : \mathbf{R} \to \mathbf{R}_{a,+\infty}$,

defined by: for every $x\in\mathbf{R}$,

$$h(x)=a+e^x, \tag{3.3.1‴}$$

is an isomorphism between the two fields, in which case the function:

$h^{-1} : \mathbf{R}_{a,+\infty} \to \mathbf{R}$,

defined by: for every $x\in\mathbf{R}_{a,+\infty}$,

$$h^{-1}(x)=\ln(x-a), \tag{3.3.2‴}$$

is a field isomorphism from the field $(\mathbf{R}_{a,+\infty},\Delta,\perp)$ to the field $(\mathbf{R},+,\cdot)$.

d) We consider the following diagram:

$$\begin{tikzcd}
 & (\mathbf{R}_{-a,a},\circ,*) \arrow[dr, "v"] & \\
(\mathbf{R}_{-\infty,-a},\bigcirc,\bullet) \arrow[ur, "u"] \arrow[rr, "w=v\circ u"] & & (\mathbf{R}_{a,+\infty},\Delta,\perp) \\
 & (\mathbf{R},+,\cdot). \arrow[ul, "f"] \arrow[uu, "g"] \arrow[ur, "h"] &
\end{tikzcd} \tag{3.10.1}$$

Then, according to the proof of Application 3.5.1, the function:

$u : \mathbf{R}_{-\infty,-a} \to \mathbf{R}_{-a,a}$,

defined by: for every $x\in\mathbf{R}_{-\infty,-a}$,

$$u(x)=(g\circ f^{-1})(x)=\frac{a(a+1+x)}{a-1+x}, \tag{3.5.2‴}$$

is a field isomorphism from the field $(\mathbf{R}_{-\infty,-a},\bigcirc,\bullet)$ to the field $(\mathbf{R}_{-a,a},\circ,*)$, and according to the proof of Application 3.7.1, the function:

$v : \mathbf{R}_{-a,a} \to \mathbf{R}_{a,+\infty}$,

defined by: for every $x\in\mathbf{R}_{-a,a}$,

$$v(x)=(h\circ g^{-1})(x)=\frac{x(a-1)+a+a^2}{a+x}, \tag{3.7.2‴}$$

is a field isomorphism from the field $(\mathbf{R}_{-a,a},\circ,*)$ to the field $(\mathbf{R}_{a,+\infty},\Delta,\perp)$. Because the composed of two field isomorphisms is also a field isomorphism, it follows that, the function:

$w : \mathbf{R}_{-\infty,-a} \to \mathbf{R}_{a,+\infty}$, defined by: for every $x\in\mathbf{R}_{-\infty,-a}$,

$$w(x)=(v\circ u)(x)$$

$$=\frac{a^2-1+ax}{a+x},\qquad (3.10.2)$$

is a field isomorphism from the field $(\mathbf{R}_{-\infty,-a},\circ,\bullet)$ to the field $(\mathbf{R}_{a,+\infty},\Delta,\perp)$.

The isomorphism w can be obtained from the proof of Application 3.9.1, where it follows that, for every $x\in\mathbf{R}_{-\infty,-a}$:

$$w(x)=(h\circ f^{-1})(x)$$

$$=\frac{a^2-1+ax}{a+x}.\qquad (3.10.2)$$

Now, all four statements in the statement are fully proven. □

The following remark are required here:

Remark 3.10.2: *In Application 3.10.1 the functions f and h are increasing, and the function h is decreasing. Observe that we can also consider case in which one of the functions f or h (or even both) be descending, that is f to be of the type of (3.1.1′) or h to be of the type of (3.3.1′), and the function g to be of the type of (3.2.1′); the conclusion of Application 3.10.1 remaining the same.* □

The inductive approach of this isomorphism of fields involves solving the problem:

Problem 3.10.3 *We consider $a\in(0,+\infty)$, and the following two laws of composition:*

$„\circ": \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *and* $„\bullet": \mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

$„\circ": \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *and* $„*": \mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

respectively

$„\Delta": \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *and* $„\perp": \mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

defined by: for every $(x,y)\in\mathbf{R}\times\mathbf{R}$,

$x\circ y=-a-a^2-a(x+y)-xy$ *and* $x\bullet y=-a-(-a-x)^{-\ln(-a-y)}$,

$$x\circ y=\frac{a^2(x+y)}{xy+a^2}\quad\text{and}\quad x*y=\frac{a-a\left(\frac{a-x}{x+a}\right)^{\ln\left(\frac{a-y}{y+a}\right)}}{1+\left(\frac{a-x}{x+a}\right)^{\ln\left(\frac{a-y}{y+a}\right)}},$$

respectively

$x\Delta y=a+a^2-a(x+y)+xy$ *and* $x\perp y=a+(x-a)^{\ln(y-a)}$.

Prove that:

a) *the maps „○" and „•" determines on the set:*

$\mathbf{R}_{-\infty,-a}=(-\infty,a)$

a structure of commutative field and that the field $(\mathbf{R}_{-\infty,a},\circ,\bullet)$ is isomorphic to the field $(\mathbf{R},+\cdot)$;

b) *the maps „∘" and „ *" determines on the set:*

$\mathbf{R}_{-a,a}=(-a,a)$

a structure of commutative field and that the field $(\mathbf{R}_{-a,a},\circ,*)$ is isomorphic to the field $(\mathbf{R},+\cdot)$;

c) *the maps „Δ" and „⊥" determines on the set:*

$\mathbf{R}_{-a,a}=(-a,a)$

a structure of commutative field and that the field $(\mathbf{R}_{-a,a},\circ,*)$ is isomorphic to the field $(\mathbf{R},+\cdot)$;

$\mathbf{R}_{a,+\infty}=(a,+\infty)$

d) *the fields $(\mathbf{R}_{-\infty,a},\circ,\bullet)$, $(\mathbf{R}_{-a,a},\circ,*)$ and $\mathbf{R}_{a,+\infty}=(a,+\infty)$ are isomorphic.*

Hint: As with Problem 3.1.3, Problem 3.2.3, respectively Problem 3.3.3, it shows that the statements in points a), b), repectively c) hold. To determine the isomorphism required in point c) reconstitute the diagram (3.10.1). □

Conclusions:-

Therefore, we have proven that there are bijective functions f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f'_5 and f'_7 such that the following diagram represent a diagram of isomorphic algebraic structures (of the same type, i.e. groupoids, semigroups, monoids, groups, rings or fields) and to be commutative:

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5-32

$$
\begin{tikzcd}
\mathbf{R}_{-\infty,a} \arrow[rr, "f_9"] \arrow[dd, "f_4"] & & \mathbf{R}_{d,+\infty} \arrow[dd, "f_8"'] \\
& \mathbf{R} \arrow[ul, "f_1"'] \arrow[ur, "f_3"] \arrow[d, "f_2"'] & \\
\mathbf{R}_{-\infty,m} \arrow[r, "f_5"] \arrow[dr, "f'_5"'] & \mathbf{R}_{b,c} \arrow[r, "f_7"] \arrow[d, "f_6"'] & \mathbf{R}_{q,+\infty} \arrow[dl, "f'_7"] \\
& \mathbf{R}_{n,p} &
\end{tikzcd}
$$

This shows that everything you can build on the set **R** of real numbers can be built on any open, bounded or unlimited interval **R**.

We consider that we have achieved the intended purpose, presented at the end of the Introduction and at the same time we have opened the way to new themes / issues of this type.

Finally, mention that we wrote this work in the context of a personal culture of teaching - learning Mathematics, culture developed and developed through individual study and experimentation in class.

In this way we used, for the theoretical part (pedagogy, psychology) the works: [02], [04], [05], [06], [07], [08], [10], [12], [14], and [15], and for the applied part the works: [01], [03], [09], [11] and [13].
The notations we used were those in [16].

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